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Complex Numbers

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Introduction

- Why does it exist??

Early in history, it was observed that some of equations do not have any real solution (real number) such as:

$$x^2 + 1 = 0$$

$$x^2 - 3x + 5 = 0$$

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Cartesian Form

- A complex number, denoted as *z*, is consisted of two kinds of number: real and imaginary. This is the traditional form called cartesian form.

$$z = x + iy$$

- As seen in the equation, the imaginary part is followed by *i*. *i* is such an imaginary number that:

$$i^2 = -1$$

- e.g: $z^2 - 3z + 5 = 0 \;=>\; z = \frac{-(-3) \pm \sqrt{9 - 4(1)(5)}}{2(1)}$

$$z = \frac{3 \pm \sqrt{-11}}{2} = \frac{3 \pm 11\sqrt{-1}}{2} = \frac{3 \pm 11i}{2} = 1.5 \pm 5.5i$$

Cartesian Form

- Basic Arithmetic Operations:

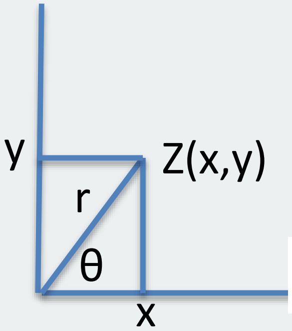
$$(x1 + i\,y1) \pm (x2 + i\,y2) = (x1 + x2) \pm i\,(y1 + y2)$$

$$(x1 + i\,y1).(x2 + i\,y2) = x1x2 + ix1y2 + ix2y1 + i^2y1y2$$

$$= (x1x2 - y1y2) + i\,(x1y2 + x2y1)$$

$$\frac{x1 + i\,y1}{x2 + i\,y2} = \frac{x1 + i\,y1}{x2 + i\,y2}.\frac{x2 - i\,y2}{x2 - i\,y2} = \frac{(x1x2 + y1y2) + i\,(y1x2 - y2x1)}{xix2 + y1y2}$$

Polar Form



A diagram used to represent complex number called an ***Argand Diagram***

The ***polar*** form is introduced by setting:

$$x = rcos\theta \;\; and \;\; y = rsin\theta$$

$$z = rcos\theta + i\,rsin\theta = r\,(cos\theta + i\;sin\theta)$$

The value *r* is called the ***absolute value*** or the ***modulus of z*** and is denoted as **|z|**.

$$|z| = r = \sqrt{x^2 + y^2}$$

Polar Form

The directed angle θ measured from positive real axis to the line z is called the argument of z and is denoted as arg z. It is measured in radians and is positive in counterclockwise sense.

$$\arg z = \theta = sin^{-1}(\frac{y}{r}) = cos^{-1}\left(\frac{x}{r}\right) = tan^{-1}(\frac{y}{x})$$

Conjugate

The conjugate of a complex number z = x + iy; denoted as z*, is defined as:

$$z = x + iy <-> \quad z^* = z - iy$$

Some properties involving conjugate are:

$(z^*)^* = z$

$|z^*| = |z|$

$z + z^* = 2Re(z)$

$zz^* = |z|^2$

$z - z^* = i\,[2Im(z)]$

$$\frac{z1}{z2} = \frac{z1z2^*}{|z2|^2}$$

Operation in Polar Form

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The polar form of a complex number may also be written in its corresponding exponential form:

$$z = r\,(\cos\theta + i\sin\theta) = re^{i\theta}$$

From this characteristic, the following equations can be made of:

$$z1 = r1\,(\cos\theta + i\sin\theta) = r1e^{i\theta}$$

$$z2 = r2\,(\cos\theta + i\sin\theta) = r2e^{i\theta}$$

Then, (cont.d)

Operations in Polar Form

(Cont.d)

$$z1\, z2 = \left(r1e^{i\theta 1}\right)\left(r2e^{i\theta 2}\right) = \; r1r2e^{i(\theta 1+\, \theta 2)}$$

$$z1\, z2 = \; r1r2[\cos(\theta 1 + \; \theta 2) + i \sin(\theta 1 + \; \theta 2)]$$

$$\frac{z1}{z2} = \frac{r1\, e^{i\theta 1}}{r2\, e^{i\theta 2}} = \frac{r1}{r2} e^{i(\theta 1-\, \theta 2)}$$

$$\frac{z1}{z2} = \frac{r1}{r2} [\cos(\theta 1 - \; \theta 2) + \; i \sin(\theta 1 - \; \theta 2)]$$

Operations in Polar Form

From Euler's Formula we can find Moivre's theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}$$

$$z^n = r^n\,(\cos n\theta + i\sin n\theta)$$

From this, we can get:

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \& \quad z^n - \frac{1}{z^n} = 2\sin n\theta$$

Derived from:

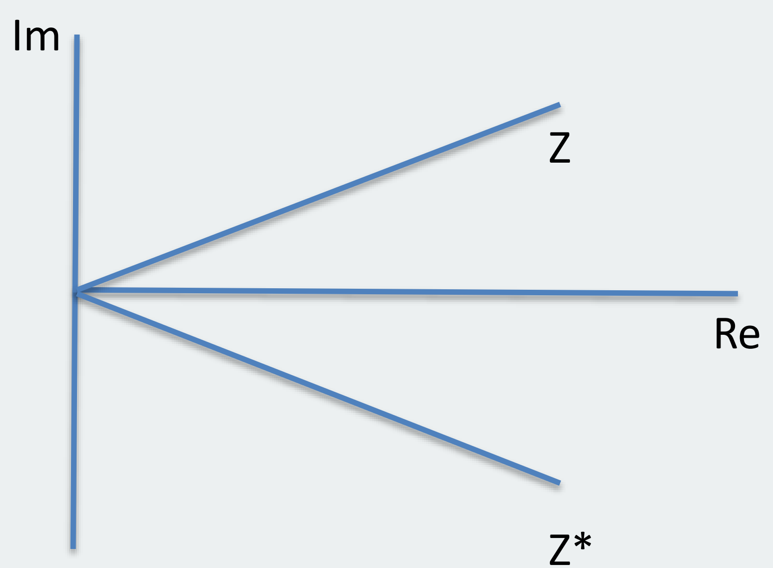
$$z + \frac{1}{z} = 2\cos\theta \quad \& \quad z - \frac{1}{z} = 2\sin\theta$$

Geometrical Effects

In the polar form, the conjugate of $z = r(\cos\theta + i\sin\theta)$ is:

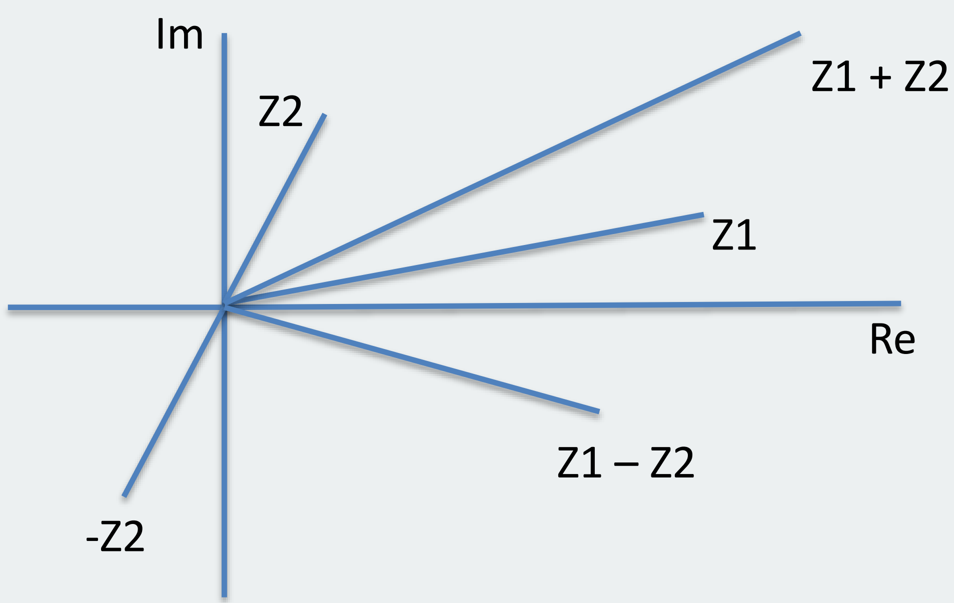
$$z^* = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) - i\sin(-\theta))$$

It follows that the point z* is the reflection of point z in the *Re-axis*:



Geometrical Effects

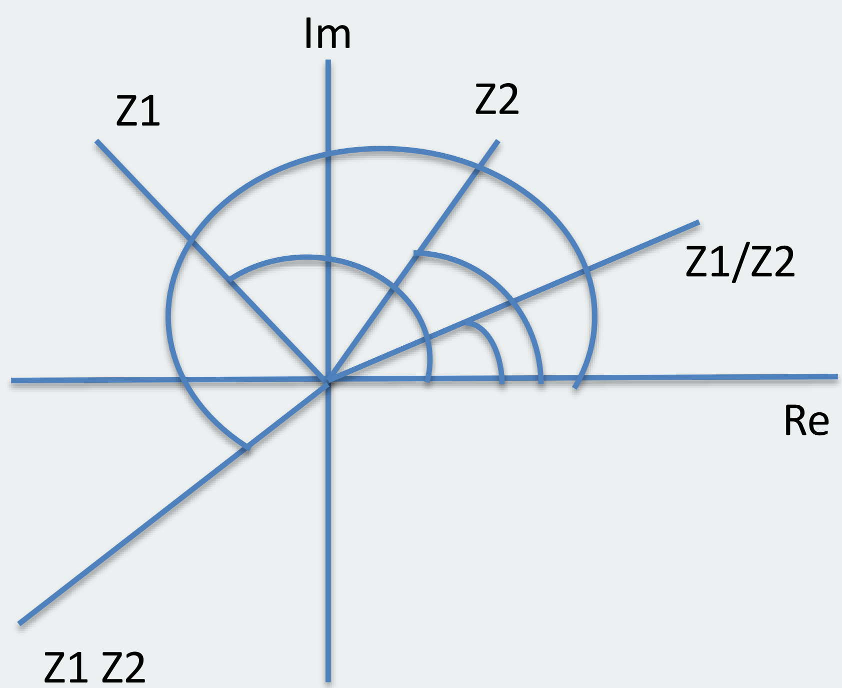
Geometrically, the addition and subtraction of 2 complex numbers are in accordance with the vector parallelogram that represents the addition and subtraction of vectors.



Geometrical Effects

The geometrical effects of multiplication and division of complex numbers can be deduced from:

- $|Z1\ Z2| = |Z1|\ |Z2|$
- $|Z1/Z2| = |Z1|/|Z2|$
- arg (Z1 Z2) = arg (Z1) + arg (Z2)
- arg (Z1/Z2) = arg (Z1) – arg (Z2)



Geometrical Effects

The imaginary factor *i* and the real number **1** are also points in the complex plane, and they have following properties:

$$|i| = 1 \; and \; \arg(i) = \frac{1}{2}\pi$$

$$|1| = 1 \; and \; \arg(1) = 0$$

For any complex number *z*:

$$|i\, z| = |i||z| = |z|$$

$$\arg(i\, z) = \arg(i) + \arg(z) = \frac{1}{2}\pi + \arg(z)$$

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}$$

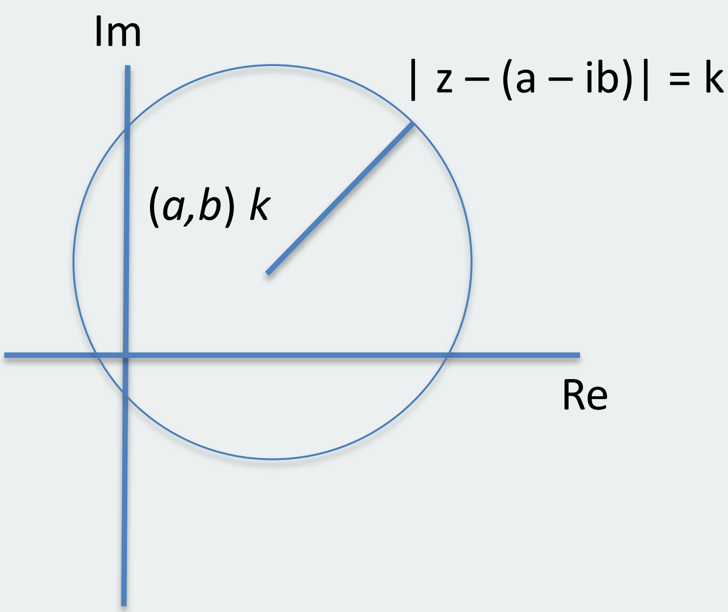
$$\arg\left(\frac{1}{z}\right) = \arg(1) - \arg z = -\arg z$$

Loci

- Locus is a set of points whose location satisfied or is determined by one or more specified conditions. In this complex numbers, there are several loci that can be classified in equations of complex such as:
  - Circular Loci
  - Linear Loci
  - Half Line
  - Sector Loci
  - Vertical and Horizontal Loci

Loci

- Circular Loci



$$|z-(a+ib)|=k$$

Represents a **circle centered** at (a, b), ***radius*** = ***k***. If a and b are 0, it is centered at the origin.

$$|z-(a-ib)|\le k$$

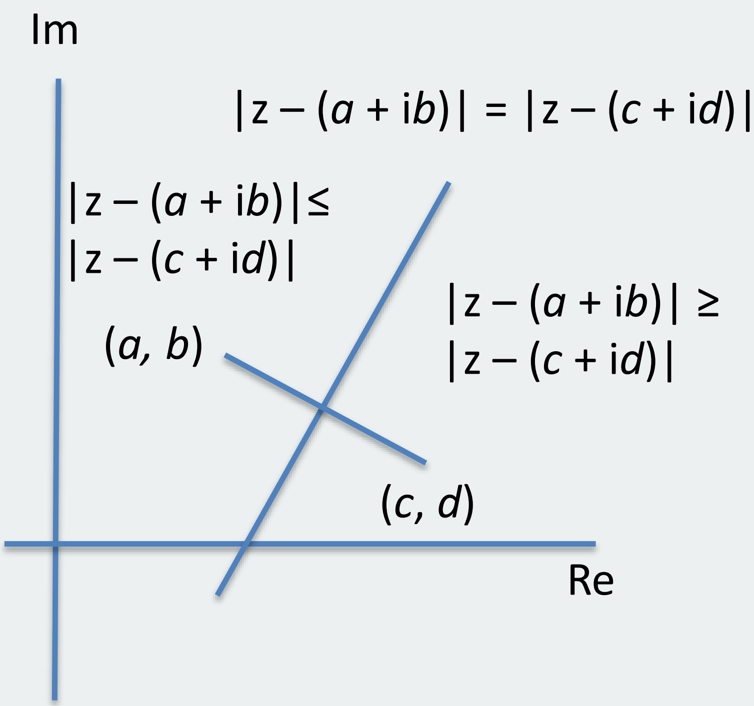
Represents a **circular region** including the circumference centered at (a, b), ***radius*** = ***k.***

$$|z-(a-ib)|\ge k$$

Represents a similar **circular region** excluding the circumference.

Loci

- Linear Loci



$$|\, z-(a+ib)\,| = |\, z-(c+id)\,|$$

Represents the **perpendicular bisector** of the line segment joining the points *(a,b)* and *(c, d)*.

$$|\, z-(a+ib)\,| \leq |\, z-(c+id)\,|$$

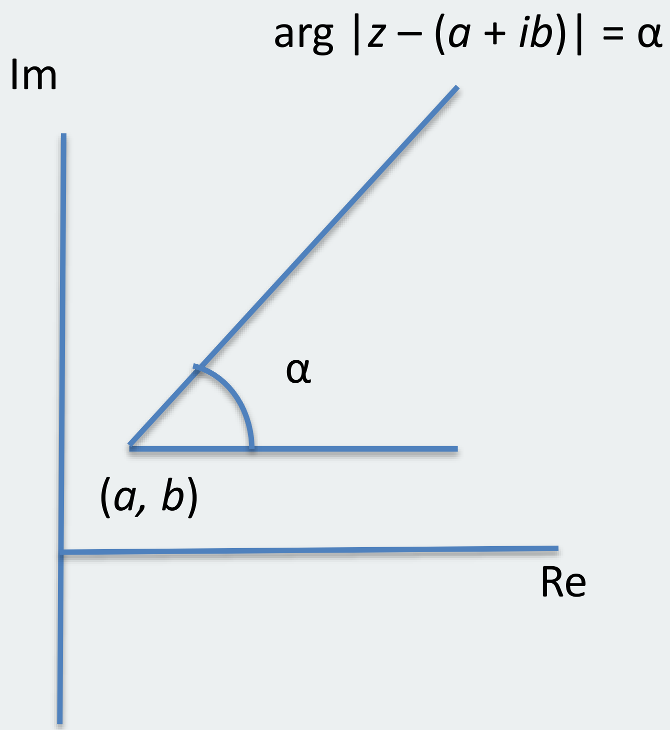
Represents the **region on the side of (*a*, *b*)** of the perpendicular bisector.

$$|\, z-(a+ib)\,| \geq |\, z-(c+id)\,|$$

Represents the **region on the side of (*c*, *d*)** of the perpendicular bisector.

Loci

- Half Line

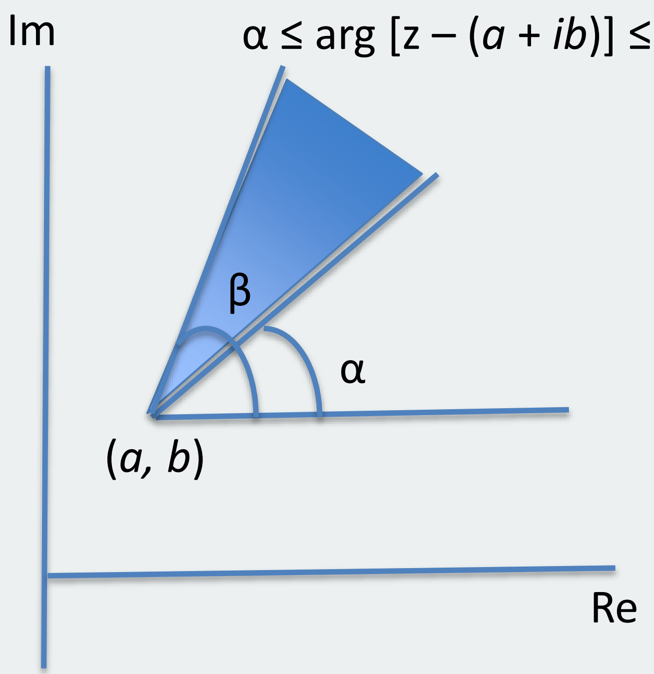


arg $|z - (a + ib) = \alpha$

Represents a half-line starting from (a, b) making an angle α with the positive direction of the Re-axis.

Loci

- Sector Loci

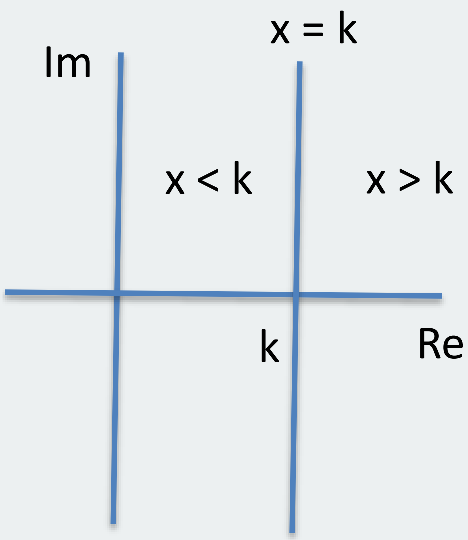


$\alpha \le \arg[z - (a + ib)] \le \beta$

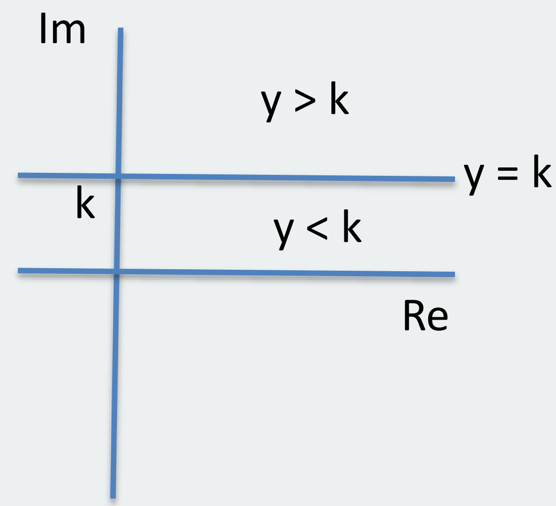
Represents a region within sector, centered at (a, b), from angle α to β made with the positive direction of the Re-axis

Loci

- Vertical and Horizontal Loci



Vertical Line



Horizontal Line