MH1810 Math 1 Part 3 Differentiation

Inverse Functions

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Derivative of Inverse Function

We state without proof the result on the derivative of inverse function.

Theorem (Derivative of inverse)

If f is increasing (respectively decreasing) and continuous on an interval (a, b) and $f'(x_0) > 0$ (respectively $f'(x_0) < 0$) for some $x_0 \in (a, b)$, then f^{-1} is differentiable at the point $y_0 = f(x_0)$, and

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

Note that the condition that f is increasing and continuous on (a, b) tells us that the function f is injective and the inverse f^{-1} exists.
In the next example, we demonstrate the formula

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

Example

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Let $f(x) = \cos x$, where $x \in (0, \pi)$. Find $(f^{-1})'(0)$.

Solution

Note that

- $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))}$

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Solution

Note that

- $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))}$
- $= \dfrac{1}{-\sin(\cos^{-1}(0))} = \dfrac{1}{-\sin(\pi/2)} = -1.$

Inverse Trigonometric Functions

Theorem

1. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$, *for* $-1 < x < 1$.
2. $\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$, *for* $-1 < x < 1$.
3. $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$, *for* $x \in \mathbb{R}$.
4. $\dfrac{d}{dx}\left(\cot^{-1} x\right) = \dfrac{-1}{1+x^2}$, *for* $x \in \mathbb{R}$.
5. $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$, *for* $x < -1$ *or* $x > 1$.
6. $\dfrac{d}{dx}\left(\csc^{-1} x\right) = \dfrac{-1}{|x|\sqrt{x^2-1}}$, , *for* $x < -1$ *or* $x > 1$.

Proof of (1).

We use implicit differentiation to obtain the derivative of the inverse function.

Let $y = f(x) = \sin^{-1}(x)$, where $x \in (-1, 1)$ and $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Note that

$$y = f(x) = \sin^{-1}(x) \iff \sin y = x.$$

Differentiate $\sin y = x$ with respect to x implicitly, we have

$$(\cos y)\,\frac{dy}{dx} = 1, \text{ which gives } \frac{dy}{dx} = \frac{1}{\cos y}.$$

□

Proof of (1) (Cont'd).

We have: $\frac{dy}{dx} = \frac{1}{\cos y}$.

By the trigonometric identity $\cos^2 y + \sin^2 y = 1$, we have

$$\cos^2 y = 1 - \sin^2 y = 1 - x^2.$$

Since $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\cos y > 0$. Therefore, $\cos y = \sqrt{1 - x^2}$.Thus, we have

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}.$$

$\square$

Inverse Trigonometric Functions

The derivatives of the other inverse trigonometric functions can be proved similarly. You should at least verify (via implicit differentiation)

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$$