MH1810 Math 1 Part 3 Differentiation

Rate of Change

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Problems on Rate of Change

For a function f, the derivative $f'(x)$ can be interpreted as the instantaneous rate of change of $f(x)$ with respective to x.

Applications:

- Physics: velocity, acceleration, electricity, ...
- Chemistry: rate of reaction
- Biology: population growth
- Economics: concepts of marginalism
- Engineering: many....

Example

Example

The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$, where the radius r is measured in micrometers ($1\mu\text{m} = 10^{-6}\text{m}$).

(a) Find the average rate of change of V with respect to r when r changes from 5 to 8 μm.

(b) Find the instantaneous rate of change of V with respect to r when $r = 5\mu$m.

Solution

Consider the function $V(r) = \frac{4}{3}\pi r^3$, where $r > 0$.

(a) When r changes from 5 to 8, we have V changes from $V(5)$ to $V(8)$. The average rate of change of V with respect to r when r changes from 5 to 8 μm is

$$\frac{V(8) - V(5)}{8 - 5} = \frac{4\pi}{3} \cdot \frac{8^3 - 5^3}{8 - 5} = 129.$$

(b) Note that $V'(r) = 4\pi r^2$.
The instantaneous rate of change of V with respect to r when $r = 5$ is $V'(5) = 4\pi(5^2) = 100\pi$.

Example

If a ball is given a push so that it has an initial velocity of 5m/s down a certain inclined plane, then the distance it has rolled after t seconds is $x = 5t + 3t^2$.

(a) Find the velocity after 2s.

(b) How long does it takes for the velocity to reach 35m/s?

(c) What is the acceleration after 2s?

(a) To find the velocity after 2s, we evaluate $x'(2)$.
Note that $x'(t) = 5 + 6t$. Hence, we have $x'(2) = 17$m/s.

(b) The find time for the velocity to reach 35m/s, we solve $x'(t) = 5 + 6t = 35$, which gives $t = 5$s.

(c) The acceleration after ts is given by $x''(t) = 6$m/s ^{2}, which is also the acceleration after 2s.

Example - Independent Reading

Example

When air expands without losing or gaining heat, its pressure P and volume V satisfy the equation

$$PV^{1.4} = C,$$

where C is a constant. At one instant the volume is $400\,\mathrm{cm}^3$, the pressure is $80\,\mathrm{kPa}$ and the pressure is decreasing with $10\,\mathrm{kPa/min}$.
At what rate is the volume increasing at this instant?

Solution - Independent Reading

Solution

Given $PV^{1.4} = C$. *Since both* V *and* P *depend on time* t, *differentiating the equation above (implicitly) with respect to* t, *we get*

$$\frac{dP}{dt} \cdot V^{1.4} + P \cdot 1.4V^{0.4} \cdot \frac{dV}{dt} = 0.$$

Solving for $\frac{dV}{dt}$ *we have*

$$\frac{dV}{dt} = -\frac{\frac{dP}{dt} \cdot V^{1.4}}{1.4PV^{0.4}} = -\frac{\frac{dP}{dt} \cdot V}{1.4P}.$$

Substituting $V = 400\,\mathrm{cm}^3$, $P = 80\,\mathrm{kPa}$ *and* $dP/dt = -10\,\mathrm{kPa}/\mathrm{min}$ *into this gives us,*

$$\frac{dV}{dt} = -\frac{\frac{dP}{dt} \cdot V}{1.4P} = -\frac{-10(400)}{1.4(80)}\,\mathrm{cm}^3/\mathrm{min} \approx 36\,\mathrm{cm}^3/\mathrm{min},$$

Example

Example

A motorcyclist is travelling along a road. On an overhead bridge is a traffic police with a radar gun, positioned 6 m above the motorcycle. When the motorcyclist is passing a lamppost, the traffic police knows that the distance from the motorcycle to the bridge is 8 m. At this moment, his instrument tells him that the distance y (in km) between him and the motorcycle is decreasing at a rate of 52 km/h. If the speed limit on this particular road is 60 km/h, can the motorcyclist be fined?

Solution

Let x (in km) be the distance from the motorcycle to the (bottom of the) bridge. As the motorcycle is moving, both x and y change, $x = x(t)$, $y = y(t)$. They are all the time however, related by the equation

$$x(t)^2 + (0.006)^2 = y(t)^2.$$

Differentiating this equation with respect to t, we get

$$2x(t)x'(t) + 0 = 2y(t)y'(t), \textit{ which gives } x'(t) = \frac{y(t)y'(t)}{x(t)}.$$

Solution (cont'd)

Solution (cont'd)

We have shown

$$x'(t) = \frac{y(t)y'(t)}{x(t)}.$$

At the moment in time $t = t_0$ *when the motorcycle passes the lamppost, note that* $y'(t_0) = -52$ *km/h*, $x(t_0) = 8\,\mathrm{m} = 0.008\,\mathrm{km}$ *and* $y(t_0) = 10\ \mathrm{m} = 0.01\,\mathrm{km}$. *So, we have*

$$x'(t_0) = \frac{(0.01)(-52)}{0.008}\,\mathrm{km/h} = \frac{(-10)52}{8}\,\mathrm{km/h} = -65\,\mathrm{km/h}.$$