MH1810 Math 1 Part 3 Differentiation

Linearization

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Linearization

Aim: To approximate $f(x)$ near $x = a$ by a linear function $L(x)$ through $x = a$.

Definition

The linearization of f at a is the linear function

$$L(x) = f(a) + (x - a)f'(a)$$

Diagram to illustrate:

Remark

(a) Note that the equation $y = L(x)$, i.e.,

$$y = f(a) + f'(a)(x - a)$$

is the equation of tangent of the curve $y = f(x)$ at $x = a$.

(b) Linearization is a local approximation for f at a via the tangent of $y = f(x)$ at $x = a$.
We use the linearization $L(x)$ to approximate value of $f(x)$ for x near a, i.e., $f(x) \approx L(x)$.

Remark

(c) (OPTIONAL.) There is also quadratic approximation of f at $x = a$. More generally, if f can be differentiated n times at $x = a$, we have the Taylor's polynomial $P_n(x)$ (degree n) of $f(x)$ at $x = a$, .

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)$$

$$+ \cdots + \frac{(x-a)^k}{k!}f^{(k)}(a)$$

$$+ \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a).$$

Example

Example

Consider $f(x) = \ln(1+x)$. Use the linearization of f at $a = 0$ to approximate the value of $\ln(1.01)$.

Solution

Solution

The linearization of f is given by $L(x)$ as follows:

$$L(x) = f(0) + f'(0)(x - 0).$$

Since $f'(x) = \dfrac{1}{1+x}$, *we have* $f'(0) = 1$ *and* $f(0) = \ln 1 = 0$ *so that*

$$L(x) = x.$$

We approximate $\ln(1.01) = \ln(1 + 0.01)$ *using linearization,*

$$f(0.01) = \ln(1.01) \approx L(0.01) = 0.01.$$

Example

Example

Use the linearization of $f(x) = \sqrt{x}$ at $a = 4$ to approximate the value of $\sqrt{4.001}$.

Solution

Let $f(x) = \sqrt{x}$, *we have* $f'(x) = \frac{1}{2\sqrt{x}}$.
The linearization of f *at* $a = 4$ *is:*

$$L(x) = f(4) + f'(4)(x-4) = \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4) = 2 + \frac{1}{4}(x-4).$$

Thus

$$\sqrt{4.001} = f(4.001) \approx L(4.001) = 2 + \frac{1}{4}(4.001-4)$$

$$= 2 + \frac{1}{4}(0.001) = 2.00025.$$

Example

Example

Approximate $\sqrt[3]{7.99}$ by linearization

$$L(x) = f(a) + f'(a)(x - a)$$

Question: Which function $f(x)$ and at which point a would you choose?

Estimation of Change using Differentials

Consider a differentiable function f, suppose the value of x changes from $x = a$ to $x = a + dx$ (i.e., $\Delta x = dx$). (Here, we have used the symbol dx to denote Δx.)

Then the corresponding change in f is

$$\Delta f = f(a + dx) - f(a).$$

Differentials

When the change dx is small, we can approximate the change in f using the its linearization at $x = a$. We denote by df the change in this linearization:

$$df = L(a + dx) - L(a) = f'(a)dx$$

The quantity df is called the differential of f.
Again, this approximation is good for values of x close to a.

Remarks

(a) Warning The derivative $\dfrac{df}{dx}$ is not the quotient the differential df and the change dx.

(b) It is useful to take note of other types of change for a change dx in x:

	Actual Change	Estimated change
Absolute change	Δf	df
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$100\frac{\Delta f}{f(a)}$	$100\frac{df}{f(a)}$

Example

Example

The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.

(a) Use the differentials to estimate the maximum error in the calculated area of the disk.

(b) What is the relative error? What is the percentage error?

Solution

Let the radius be r cm. Then the area of the circular disk is $A(r) = \pi r^2 cm^2$. Note that $A'(r) = 2\pi r$ and the differential of A at $r = r_0$ is

$$dA = A'(r_0)dr.$$

Solution (cont'd)

We have

$$dA = A'(r_0)dr.$$

(a) To use the differentials to estimate the maximum error in the calculated area of the disk, note that the differential of $A(r)$ at $r_0 = 24$ is given by $A'(24)dr$, with $dr = 0.2$.
Thus, we have the maximum error (ΔA) is estimated to be $dA = A'(24)dr = 2\pi(24)(0.2) = 9.6\pi$.
(b) The relative error is estimated to be

$$\frac{dA}{A(24)} = \frac{2\pi(24)(0.2))}{\pi(24)^2} = \frac{1}{60} \approx 0.01667.$$

The estimated percentage error is (100%) (relative error) which is $100(0.01667)\% = 1.667\%$.