MH1810 Math 1 Part 3 Differentiation

Newton's Method

Tang Wee Kee

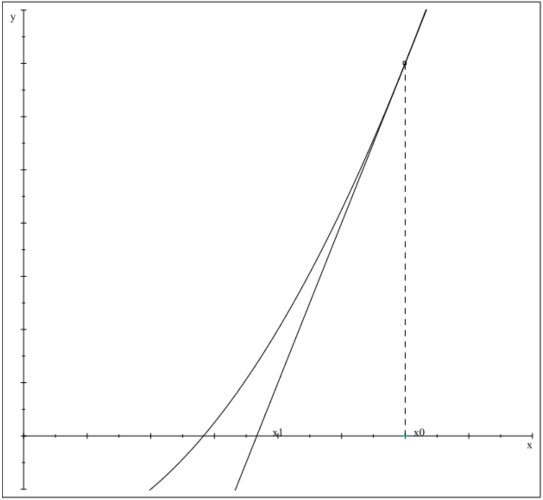
Nanyang Technological University

Newton's Method

Newton's Method is a numerical method to approximate the solution to an equation $f(x) = 0$. It is an iteration method which involves a sequence of approximations that approach the solution. It is also known as Newton-Raphson method.

The underlying idea is to approximate the graph of f by a suitable tangent line.

Idea of Newton's Method



Iteration: Steps 0 and 1

- Let x_0 be an approximate value of the root.
- Set x_1 to be the x-intercept of the tangent to the curve of f at x_0. The equation of the tangent of the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is $y = f'(x_0)(x - x_0) + f(x_0)$. Therefore, we have

$$0 = f'(x_0)(x_1 - x_0) + f(x_0)$$

It follows that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Iteration: Step (n+1)

If x_n has been found, let x_{n+1}be the x-intercept of the tangent to the curve of f at x_n. Like x_1, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This iterative method of approximating the solution of $f(x) = 0$ is called Newton's method.

Newton's Method - In Summary, to find the a root of f(x)=0:

Step 0: Select x_0 as an approximate value of the root.

Subsequent steps: Evaluate Newton's iterates

$$x_{n+1} = x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)}.$$

Example

Example

Use Newton's method to solve $x^3 - x + 1 = 0$

Solution

Let $f(x) = x^3 - x + 1$, *which is continuous and differentiable with* $f'(x) = 3x^2 - 1$.
Note that $f(-1) = 1 > 0$ *and* $f(-2) = -5 < 0$.
By the Intermediate Value Theorem, $f(c) = 0$ *for some* $c \in (-2, -1)$.
Thus we may choose either $x_0 = -1$ *or* $x_0 = -2$.

Solution

Solution

Set $x_0 = -1$. *For* $n = 1, 2, 3, \ldots,$ *we have*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n + 1}{3x_n^2 - 1}.$$

Thus,

$$x_1 = x_0 - \frac{x_0^3 - x_0 + 1}{3x_0^2 - 1} = -1 - \frac{1}{2} = -1.5.$$

Solution (cont'd)

Solution (cont'd)

$$
\begin{aligned}
x_2 &= x_1 - \frac{x_1^3 - x_1 + 1}{3x_1^2 - 1} \\
&= -1.5 - \frac{(-1.5)^3 - (-1.5) + 1}{3(-1.5)^2 + 1} \\
&= -1.34883.
\end{aligned}
$$

and so on.

Tabulation of Iterates

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
0	-1	1	2	-1.5
1	-1.5	-0.87	5.75	-1.34783
2	-1.34783	-0.10068	4.449905	-1.32520
3	-1.32520	-0.002058	4.264635	-1.324718
4	-1.324718	$-9.2E-07$	4.264633	-1.324718

An approximated root, to five decimal places, for $x^3 - x + 1 = 0$ is

$$x_4 = x_5 = -1.324718.$$

Other uses of Newton's Method

A classical example of the use of Newton's method is to compute values of non-polynomial quantities like the reciprocal of a number.

Example

Use Newton's method to find an approximate value the reciprocal of α, where $\alpha \neq 0$.

Approximating Reciprocals

Our aim: Given $\alpha \neq 0$, estimate x where $x = \dfrac{1}{\alpha}$.

Note that $x = \frac{1}{\alpha}$ is equivalent to $\frac{1}{x} = \alpha$, i.e., $\frac{1}{x} - \alpha = 0$.
Thus, we use the function

$$f(x) = \frac{1}{x} - \alpha.$$

Then the root of the equation $f(x) = 0$ is $\frac{1}{\alpha}$.

Approximating Reciprocals

By Newton's Method, we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{\frac{1}{x_n} - \alpha}{-\frac{1}{x_n^2}} \\ &= x_n + x_n^2 \left(\frac{1}{x_n} - \alpha \right) \\ &= x_n + x_n (1 - \alpha x_n) \\ &= x_n (2 - \alpha x_n) \end{aligned}$$

Note that the computation of x_{n+1} from x_n involves only addition and multiplication, which are much easier to handle than division by the earlier computers.

Approximating Square Roots

Another classical example for Newton's method is the computation of the square-root $\sqrt{\alpha}$ of a positive number α.

Example

Use Newton's method to find an iteration to approximate the positive square root of α, where $\alpha > 0$.

Solution

Note that the root of $x = \sqrt{\alpha}$ *is equivalent to the positive root of* $x^2 = \alpha$
i.e., $x^2 - \alpha = 0$.
Let $f(x) = x^2 - \alpha$.

Approximating Square Roots

Solution (cont'd)

By Newton's method, the iterates are

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f\left(x_n\right)}{f'\left(x_n\right)} \\
&= x_n - \frac{x_n^2 - \alpha}{2x_n} \\
&= \frac{x_n}{2} + \frac{\alpha}{2x_n} \\
&= \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).
\end{aligned}
$$

Question Can we use the function $f(x) = x - \sqrt{\alpha}$?

Example

Example

Approximate the value of $\sqrt{3}$.

Solution

We know that to approximate $\sqrt{\alpha}$*, we can use the iteration*

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

To approximate $\sqrt{3}$*, we use the above, with* $\alpha = 3$ *and* $x_0 = 2$*:*

$$x_1 = \frac{1}{2}\left(2 + \frac{3}{2}\right) = \frac{7}{4} = 1.75$$

$$x_2 = \frac{1}{2}\left(\frac{7}{4} + \frac{3}{7/4}\right) = \frac{97}{56} \approx 1.7321429$$