MH1810 Math 1 Part 3 Differentiation

Mean Value Theorem and L'Hospital Rule

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Theorem (The Mean Value Theorem)

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Mean Value Theorem - Graphical Illustration

Using Mean Value Theorem

Example

Suppose $f(0) = -3$ and $f'(x) \leq 5$ for all x, how large can $f(2)$ be?

Solution

Since f is differentiable for all x, f is also continuous everywhere. Applying the Mean Value Theorem to f on $[0, 2]$ we have for some $c \in (0, 2)$ that

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \leq 5,$$

so

$$f(2) \leq f(0) + 5(2 - 0) = -3 + 10 = 7,$$

so the largest value that $f(2)$ can have is 7.

Using Mean Value Theorem in Approximation

Example

Use the Mean Value Theorem to estimate $\sqrt[3]{65}$.

Note that $64 < 65 < 125$, where $\sqrt[3]{64} = 4$ and $\sqrt[3]{125} = 5$.
This suggests that we consider $f(x) = \sqrt[3]{x}$ where $x \in [64, 65]$.

Solution

We shall use the function $f(x) = \sqrt[3]{x}$.

Solution

The function $f(x) = \sqrt[3]{x}$ *is continuous on* $[64, 65]$ *and differentiable on* $(64, 65)$ *with*

$$f'(x) = \frac{1}{3x^{2/3}}, x \in (64, 65).$$

By Mean Value Theorem, there is an $x_0 \in (64, 65)$ *such that*

$$\frac{f(65) - f(64)}{65 - 64} = f'(x_0),$$

which gives

$$\sqrt[3]{65} - 4 = \frac{1}{3}x_0^{-2/3}.$$

Thus we have

$$\sqrt[3]{65} = 4 + \frac{1}{3x_0^{2/3}}, \text{ where } x_0 \in (64, 65).$$

Solution (Cont'd)

Solution

Next, we estimate the value $\frac{1}{3x_0^{2/3}}$. *Since* $64 < x_0 < 65$, *we have*

$$3(64^{2/3}) < 3x_0^{2/3} < 3(65^{2/3}),$$

and hence

$$\frac{1}{3x_0^{2/3}} < \frac{1}{3(64^{2/3})} = \frac{1}{3(4^2)} = \frac{1}{48}.$$

Thus, we have

$$\sqrt[3]{65} = 4 + \frac{1}{3x_0^{2/3}} < 4 + \frac{1}{48}.$$

Solution (Cont'd)

From the above, we have

$$4 < \sqrt[3]{65} < 4 + \frac{1}{48}.$$

We can take a number in $(4, 4 + \frac{1}{48})$ as an approximation of $\sqrt[3]{65}$.

Limits of fractions, where either both the numerator and the denominator tend to zero, or they both tend to $\pm\infty$, are called indeterminate forms (of type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ respectively).

Examples

Which of the following limits are of indeterminate form?

(a) $\lim_{x \to 1} \frac{\ln x}{x-1}$.

(b) $\lim_{x \to 1^+} \frac{x^3 - 1}{\sqrt{x-1}}$.

(c) $\lim_{x \to \infty} \frac{e^x}{x^2}$.

Such limits of indeterminate form fail to meet the requirements of the limit law

$$\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \longrightarrow a} f(x)}{\lim\limits_{x \longrightarrow a} g(x)}.$$

Many important limits are of indeterminate forms and their limits can be evaluated by the powerful result, L'Hospital's Rule.

L'Hospital's Rule

Theorem (l'Hospital's Rule)

Suppose f and g are differentiable and both $g(x)$ and $g'(x)$ are non-zero near a (except possibly at a). Suppose also that

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0,$$

or that

$$\lim_{x\to a} f(x) = \pm\infty, \quad \lim_{x\to a} g(x) = \pm\infty.$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(if the latter limit exists, but also if it diverges to ∞ or $-\infty$).

The theorem holds also for one sided limits and for limits at infinity ($x \to \pm\infty$).

Proof – Omitted.

Example

Example

Find the limit
$$\lim_{x\to 1}\frac{\ln x}{x-1}.$$

Solution

Note that $\lim_{x\to 1}\frac{\ln x}{x-1}$ *is in indeterminate form of type* '$\frac{0}{0}$'. *We can use l'Hospital's rule.*

$$\lim_{x\to 1}\frac{\ln x}{x-1}\underbrace{=}_{L'HRule}\lim_{x\to 1}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}(x-1)}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}=1.$$

What's wrong with this?

$$\lim_{x\to 1}\frac{x+1}{x} = \lim_{x\to 1}\frac{\frac{d}{dx}(x+1)}{\frac{d}{dx}x} = \lim_{x\to 1}\frac{1}{1} = 1.$$

But

$$\lim_{x\to 1}\frac{x+1}{x} = \frac{1+1}{1} = 2??$$

WARNING Note that the conditions of l'Hospital's rule must be satisfied before we can use it.

Example

Example

Evaluate the limit

$$\lim_{x\to\infty} \frac{e^x}{x^2}.$$

Solution

Sometimes we have to use l'Hospital repeatedly.

$$\underbrace{\lim_{x\to\infty} \frac{e^x}{x^2}}_{Type\,\frac{\infty}{\infty}} \underbrace{=}_{L'Hrule} \underbrace{\lim_{x\to\infty} \frac{e^x}{2x}}_{Type\,\frac{\infty}{\infty}} \underbrace{=}_{L'HRule} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Question

Would you apply L'Hospital's Rule to the following

$$\lim_{x\to\infty} \frac{\sqrt{x^2+2}}{\sqrt{x^2+5}}?$$

$$\lim_{x\to\infty} \frac{x^{179}+x^{178}+\cdots+x+1}{3x^{179}-2x^{178}+\cdots+3x-2}?$$

Other Indeterminate Form

Example

Evaluate the limit

$$\lim_{x \to 0^+} x \ln x.$$

It may take some rewriting before we can use l'Hospital's rule.

Solution

The limit $\lim_{x \to 0^+} x \ln x$ *is indeterminate form of type '*$0 \cdot \infty$*'. We cannot apply l'Hospital's rule as it is not in quotient of two functions. However, we may rewrite the function* $x \ln x$ *as a quotient.*

TRICK

$$x \ln x = \frac{\ln x}{1/x} \quad \textit{or} \quad x \ln x = \frac{x}{1/(\ln x)}.$$

Solution

Solution

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln x}{1/x}$$

$$\underbrace{=}_{LHrule} \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0^+} (-x) = 0. \quad (*)$$

Question What would you obtain if we do the following instead

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{x}{1/(\ln x)}?$$

Example

Example

Evaluate

$$\lim_{x \to 0^+} (x^x)$$

Solution

The limit $\lim_{x \to 0^+} (x^x)$ *is of indeterminate form of type* '0^0'.

Note that

$$x^x = \exp(\ln(x^x)) = \exp(x \ln x).$$

Thus, we have

$$\lim_{x \to 0^+} (x^x) = \lim_{x \to 0^+} \exp(x \ln x).$$

Solution

Solution

Since $\exp(x)$ *is continuous, we can interchange the order of taking limit and* $\exp(x)$*, i.e.,*

$$\lim_{x \to 0^+} \exp(x \ln x) = \exp \underbrace{\left(\lim_{x \to 0^+} (x \ln x) \right)}_{=0}.$$

From the preceding example, we have evaluated

$$\lim_{x \to 0^+} x \ln(x) = 0.$$

Therefore, we have

$$\lim_{x \to 0^+} (x^x) = \exp(0) = 1.$$