MH1810 Math 1 Part 4 Integration

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Antiderivatives

We now consider the ‘inverse problem’ to differentiation:

Given a function f, is there a function F such that $F'(x) = f(x)$?

If such a function F exists, it is called an antiderivative of f.

The process of finding $F(x)$ is called integration.

Antiderivatives

Definition

A function F is said to be an antiderivative of f on an interval (a, b) if $F'(x) = f(x)$ for all x in (a, b).

Example

(a) $\dfrac{d}{dx}(\sin x) = \cos x$ on $\mathbb{R}$:
$\sin x$ is an antiderivative of $\cos x$.

(b) $\dfrac{d}{dx}\left(x^3 - 4\sqrt{x} + 179\right) = 3x^2 - \dfrac{2}{\sqrt{x}}$ on $(0, \infty)$:
$\left(x^3 - 4\sqrt{x} + 179\right)$ is an antiderivative of $3x^2 - \dfrac{2}{\sqrt{x}}$.

General Antiderivatives

Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is any arbitrary constant.

Example

$\dfrac{d}{dx}(\sin x) = \cos x$ on $\mathbb{R}$:

$\sin x$ is an antiderivative of $\cos x$.

The most general antiderivative of $\cos x$ is $\sin x + C$.

Indefinite Integrals

Definition

The indefinite integral of f, denoted by $\int f(x)\,dx$ is the most general derivative of f.
The function f is called the integrand.

Example

$$\int \cos x \; dx = \sin x + C$$

$$\int 3x^2 - \frac{2}{\sqrt{x}} \, dx = x^3 - 4\sqrt{x} + 179 + C$$

Integration

- By integration, we mean the process of finding antiderivative or the indefinite integral
$$\int f(x)dx.$$
- By definition, we have
$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x).$$

Example

Example

Prove that $\int \frac{1}{x}\, dx = \ln|x| + C$.

Solution

It suffices for us to prove

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

For $x > 0$*, we have* $\dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln x = \dfrac{1}{x}$.

For $x < 0$*, we have* $\dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln(-x) = \dfrac{-1}{-x} = \dfrac{1}{x}$.

Therefore, we have proven that for $x \neq 0$, $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$; *which is equivalent to* $\int \frac{1}{x}\, dx = \ln|x| + C$.

Rules for Integration

Theorem (Rules for integration)

1. $$\int (f(x) + g(x))\ dx = \int f(x)dx + \int g(x)dx.$$
2. $$\int (f(x) - g(x))\ dx = \int f(x)dx - \int g(x)dx.$$
3. $$\int cf(x)dx = c \int f(x)dx.$$

Rules for Integration: Proof

We shall prove (1) by differentiating the expression on the right.

$$\frac{d}{dx}\left(\int f(x)dx + \int g(x)dx\right)$$

$$= \frac{d}{dx}\left(\int f(x)dx\right) + \frac{d}{dx}\left(\int g(x)dx\right)$$

$$= f(x) + g(x)$$

Thus, we have $\int (f(x) + g(x))\ dx = \int f(x)dx + \int g(x)dx$.

Examples (Independent Reading)

(a) $\displaystyle\int \left(2x^3 + 3x^{\frac{3}{2}}\right) dx = \frac{1}{2}x^4 + \frac{6}{5}x^{\frac{5}{2}} + C$

(b) $\displaystyle\int \left(4u^{-5} - 2\cos u + e^u\right)\ du = -u^{-4} - 2\sin u + e^u + C$

(c) $$\int \frac{(1+x^2)^2}{x^4}\ dx = \int x^{-4} + 2x^{-2} + 1\, dx$$

$$= \frac{-1}{3}x^{-3} - 2x^{-1} + x + C$$

(d) $\displaystyle\int \frac{1}{\sqrt{t}} + \frac{\pi}{\sqrt{1-t^2}}\ dt = 2\sqrt{t} + \pi\sin^{-1}(t) + C$

Examples (Independent Reading)

Example

If $f'(x) = 2x - 3$ and $f(2) = 3$, find $f(x)$.

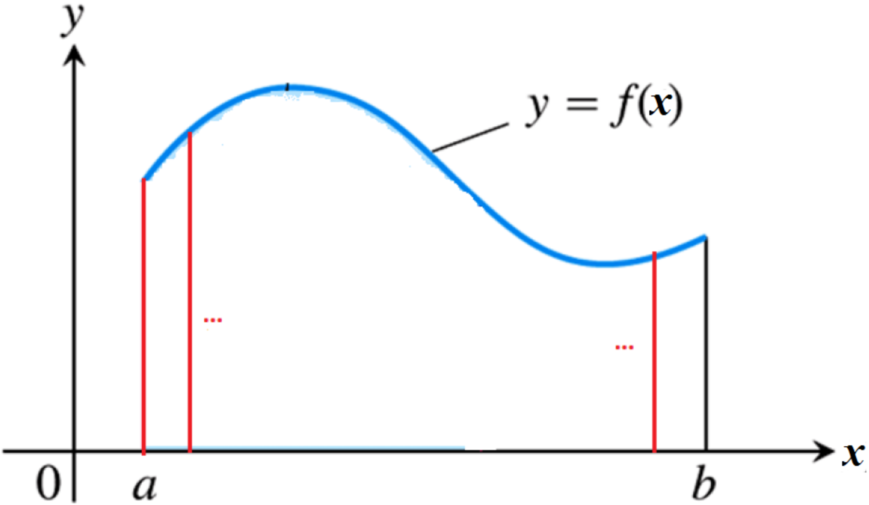
Solution

If $f'(x) = 2x - 3$, *then* $f(x) = \int 2x - 3\,dx = x^2 - 3x + C$ *for some constant* C.
Since $f(2) = 3$, *we obtain* $C = 5$.
Thus, $f(x) = x^2 - 3x + 5$.

Other Special Rules

There are integration rules correspond to the product rule and the chain rule for differentiation. These will be discussed later. They lead to special integration methods, namely **integration by parts** and **substitution rule** respectively.

Area Under a Curve $y = f(x)$, $f(x) > 0$



Area Under a Curve y = f(x), f(x) > 0

To find the area under a curve $y = f(x)$, where $f(x) > 0$ from $x = a$ to $x = b$, we divide the interval $[a, b]$ into n equal subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{k-1}, x_k], \ldots [x_{n-1}, x_n].$$

The width of each subinterval is $\Delta x = x_k - x_{k-1} = \dfrac{b-a}{n}$.
We have $x_0 = a$ and $x_n = b$. Thus, we have

$$x_k = x_0 + k\left(\frac{b-a}{n}\right) \text{ for } k = 0, 1, 2, 3, \ldots, n.$$

Area Under a Curve y = f(x), f(x) > 0

In each kth subinterval $[x_{k-1}, x_k]$, we choose a point x_k^* and evaluate the value $f(x_k^*)$. The area of the k-th rectangle, over $[x_{k-1}, x_k]$, with height $f(x_k^*)$, is

$$f(x_k^*)\Delta x = \frac{b-a}{n} f(x_k^*).$$

Now, we approximate the area under the curve $y = f(x)$ by the total areas of all these rectangles.

$$\sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*).$$

Area Under a Curve y = f(x), f(x) > 0

If the function is well behaved, as we increase the number n of subintervals, the length of the subinterval Δx tends to zero, the approximations, which are independent of how the sample points x_k^* are chosen, should approach the area A under the curve. We write this shortly as

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{b-a}{n} f\left(x_k^*\right).$$

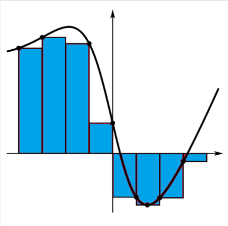
Definition

Let f be a function on $[a, b]$ and

$$x_k = a + k\left(\frac{b-a}{n}\right) \text{ for } k = 0, 1, 2, \ldots n.$$

With $x_k^* \in [x_{k-1}, x_k]$, the finite sum

$$\sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*),$$



is called a Riemann sum of f on $[a, b]$.

Example

Example

Riemann sum of $f(x) = x^2$ on $[1, 3]$.

For $k = 1, 2, 3 \ldots, n$, note that

$$x_k = 1 + k\left(\frac{3-1}{n}\right) = 1 + \frac{2k}{n}.$$

Suppose we take $x_k^* = x_k$, the right end point of the kth subinterval. We have the following Riemann sum $f(x)$ on $[1, 3]$:

$$\sum_{k=1}^{n} \frac{2}{n} f(x_k^*) = \sum_{k=1}^{n} (\frac{2}{n}) \left(1 + \frac{2k}{n}\right)^2.$$

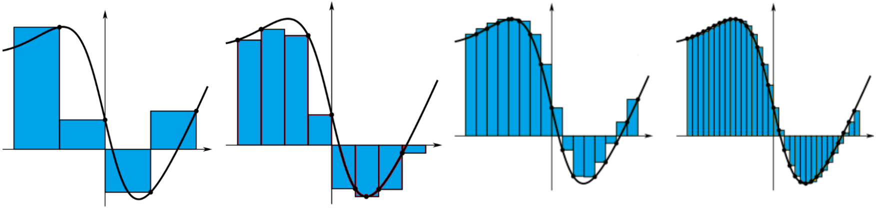
Definite Integrals

Let f be a function on $[a, b]$.

The definite integral of f from a to b, denoted by $\int_a^b f(x)\,dx$, is defined as follows

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*),$$

where the limit of the Riemann sums as $n \to \infty$ must be independent of how the sample points x_k^* are chosen.



Definite Integrals

- If $a > b$, we define
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

- If $a = b$, we define
$$\int_a^b f(x)\,dx = 0.$$

Definite Integrals

In general, the definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*),$$

may not exist.

If $\int_a^b f(x)dx$ exists, we say that f is (Riemann) integrable on $[a, b]$.

Some Riemann Integrable Functions

Theorem

If f is
(a) continuous,
(b) monotonic, or
(c) piecewise continuous with finite number of jump discontinuities
on $[a, b]$, then the definite integral $\displaystyle\int_a^b f(x)dx$ exists.

Example (Optional)

Example

Find $\int_1^3 x^2 dx$.

Solution

We partition the interval $[1, 3]$ *into n subintervals of equal width,* $\Delta x = \frac{2}{n}$*, so that* $x_k = 1 + k\Delta x = 1 + \frac{2k}{n}$*. The subintervals are*

$$[1, 1 + \frac{2}{n}], [1 + \frac{2}{n}, 1 + 2(\frac{2}{n})], \ldots, [1 + (k-1)(\frac{2}{n}), 1 + (k)(\frac{2}{n})], \ldots,$$

$$\ldots, [1 + (n-1)(\frac{2}{n}), 3]$$

Take $x_k^* = x_k = 1 + \dfrac{2k}{n}$.

Solution

Solution

$$Riemann \quad Sum = \sum_{k=1}^{n} f(x_k^*)\ \Delta x = \sum_{k=1}^{n} f(1 + \frac{2k}{n})\Delta x$$

$$= \sum_{k=1}^{n} (1 + \frac{2k}{n})^2 \cdot \frac{2}{n} = \frac{2}{n}\left(\sum_{k=1}^{n} (1 + \frac{4k}{n} + \frac{4k^2}{n^2})\right)$$

$$= \frac{2}{n}\left(\sum_{k=1}^{n} 1 + \sum_{k=1}^{n} \frac{4k}{n} + \sum_{k=1}^{n} \frac{4k^2}{n^2}\right) = \frac{2}{n}\left(n + \frac{4}{n}\sum_{k=1}^{n} k + \frac{4}{n^2}\sum_{k=1}^{n} k^2\right)$$

Solution

$$\text{Riemann Sum} = \frac{2}{n}\left(n + \frac{4}{n}\sum_{k=1}^{n} k + \frac{4}{n^2}\sum_{k=1}^{n} k^2\right)$$

$$= \frac{2}{n}\left(n + \frac{4}{n}\cdot\frac{n(n+1)}{2} + \frac{4}{n^2}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 2\left(1 + 2 + \frac{2}{n} + \frac{2}{3}(2 + \frac{3}{n} + \frac{1}{n^2})\right)$$

We have used:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution

Therefore, the definite integral

$$\int_1^3 x^2\, dx = \lim_{n\to\infty} 2\left(1+2+\frac{2}{n}+\frac{2}{3}(2+\frac{3}{n}+\frac{1}{n^2})\right) = \frac{26}{3}.$$

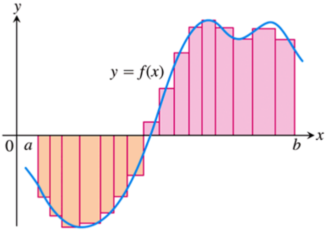
Remarks:

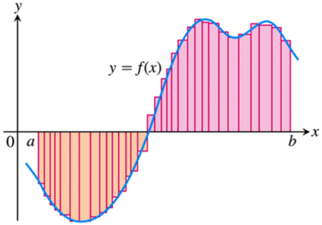
Since $x^2 \geq 0$, the value $\int_1^3 x^2\, dx = \dfrac{26}{3}$ is the area of the region under of the graph of $y = x^2$, and above the x-axis, for $1 \leq x \leq 3$.

Meaning of Definite Integral

The definite integral $\int_a^b f(x)\ dx$ is the net area between the graph of $y = f(x)$ and the x-axis.

Parts of the graph lying above (resp. under) the x-axis gives a positive (resp. negative) contribution to the area.
This is because terms where $f(x_k^*) < 0$ give a negative contribution to the Riemann sum $\sum_{k=1}^{n} f(x_k^*)\Delta x$.





Definite Integral

Suppose $f(x) \geq 0$ on $[a, b]$. The definite integral $\int_a^b f(x)\, dx$ is the area of the region bounded below by the graph $y = f(x)$ and above the x-axis, on $[a, b]$.

Lastly, the definite integral $\int_a^b f(x) dx$ is a number which is independent of the variable x.

$$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt = \int_a^b f(s)\, ds = \dots .$$

The variables x, t, s are dummy variables.

Properties of Definite Integrals

Theorem

Suppose all the definite integrals below exist. Then,

1. $\int_a^b c\,dx = c(b-a)$.
2. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.
3. $\int_a^b Kf(x)\,dx = K\int_a^b f(x)\,dx$, *where* K *is a constant.*
4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Example

Example

Evaluate $\int_1^3 (4 + x^2)\, dx$.

Solution

We have evaluated $\displaystyle\int_1^3 x^2\, dx = \frac{26}{3}$.

By property 1, $\displaystyle\int_1^3 4\, dx = 4(3-1) = 8$.

By property 2, *we have*

$$\int_1^3 (4 + x^2)\, dx = \int_1^3 4\, dx + \int_1^3 x^2\, dx$$

$$= 8 + \frac{26}{3} = \frac{50}{3}.$$

Order Preserving Property

Theorem

Suppose the following integrals exist and $a < b$.

1. $f(x) \geq 0$ *on* $[a, b] \implies \int_a^b f(x)dx \geq 0.$
2. $f(x) \geq g(x)$ *on* $[a, b] \implies \int_a^b f(x)dx \geq \int_a^b g(x)dx.$
3. $m \leq f(x) \leq M$ *on* $[a, b]$

$$\implies m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Proofs (Optional)

1. Proof of (1) follows from the definition.
2. Proof of (2) follows from (1) by applying (1) to $h(x) = f(x) - g(x)$ on $[a, b]$.
3. Proof of (3) follows from (2):
   $m \leq f(x) \leq M$ on $[a, b]$

$$\implies \underbrace{\int_a^b m\,dx}_{=m(b-a)} \leq \int_a^b f(x)\,dx \leq \underbrace{\int_a^b M\,dx}_{=M(b-a)}.$$

Example

Estimate the value of the integral $\int_1^2 \frac{1}{x}\, dx$ without evaluating it.

Solution

On the interval $[1, 2]$, *the function* $f(x) = 1/x$ *is decreasing so that its largest value occurs at the left endpoint and its smallest value at the right endpoint. So, we have*

$$\frac{1}{2} \leq f(x) \leq 1, \quad \text{for } x \in [1, 2].$$

Rough Estimates

$$\frac{1}{2} \le f(x) \le 1, \quad \text{for } x \in [1,2]..$$

By the Order-preserving property, we have

$$\frac{1}{2}(2-1) \le \int_1^2 f(x)\,dx \le 1 \cdot (2-1),$$

which means

$$\frac{1}{2} \le \int_1^2 \frac{1}{x}\,dx \le 1.$$

Even Functions

Proposition

Suppose f is an even continuous function. Then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx.$

Examples:

(a) $\int_{-5}^{5} x^2\, dx = 2\int_{0}^{5} x^2\, dx$

(b) $\int_{-\pi}^{\pi} \cos x\, dx = 2\int_{0}^{\pi} \cos x\, dx$

Odd Functions

Proposition

Suppose f is an odd continuous function. Then $\int_{-a}^{a} f(x)\,dx = 0.$

Examples:

(a) $\int_{-179}^{179} x^3\,dx = 0$

(b) $\int_{-\pi}^{\pi} \sin x\,dx = 0$