MH1810 Math 1 Part 4 Integration

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A consequence of Chain Rule for differentiation is the Substitution Rule.

Theorem

$$\int f(u(x))\,\underbrace{u'(x)\;dx}_{du} = \int f(u)du.$$

The idea behind the substitution rule is to replace a relatively complicated integral by a simpler integral.

Example

Example

Evaluate $\int \frac{x}{x^2+1}\,dx$.

[Technique:] Choose u to be some integrand whose derivative also occurs (except for a constant).

Solution

Note: $\dfrac{d}{dx}(x^2+1) = 2x$. *Let* $u = x^2+1$.

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \underbrace{\frac{1}{x^2+1}}_{\frac{1}{u}} \underbrace{(2x)\,dx}_{du} = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C = \ln\sqrt{x^2+1} + C.$$

Example

Example

Evaluate $\int \sin^3 x \cos x dx$.

Solution

Note that $\frac{d}{dx}(\sin x) = \cos x$. *Thus, we let* $u = \sin x$.

$$\int \underbrace{\sin^3 x}_{u^3} \underbrace{\cos x}_{u'} \, dx = \int u^3 \, du$$

$$= \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C$$

Example

Evaluate $\int \frac{e^{3x}}{\sqrt{1-e^{6x}}}\,dx$.

Solution

Note that $e^{6x} = (e^{3x})^2$ *and* $\dfrac{d}{dx}(e^{3x}) = 3e^{3x}$.

Let $u = e^{3x}$.

$$\int \frac{e^{3x}}{\sqrt{1-e^{6x}}}\,dx = \frac{1}{3}\int \underbrace{\frac{1}{\sqrt{1-(e^{3x})^2}}}_{1/\sqrt{1-u^2}} \underbrace{3e^{3x}\,dx}_{du}$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\sin^{-1} u + C$$

$$= \frac{1}{3}\sin^{-1}(e^{3x}) + C$$

Theorem

$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du.$

Example

Evaluate $\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\, dx$.

Solution

Choose $u = \sqrt{x+1}$, $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x+1}}$;

$$x = 0 \Leftrightarrow u = 1, x = 8 \Leftrightarrow u = 3.$$

$$\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}\, dx = \int_0^8 2(\underbrace{\cos\sqrt{x+1}}_{\cos u})\, \underbrace{\frac{1}{2\sqrt{x+1}}\, dx}_{du}$$

$$= \int_1^3 2\cos u\, du = 2\sin u\Big|_1^3 = 2(\sin 3 - \sin 1).$$

Integration by Parts

A consequence of Product Rule for differentiation is the integration by parts formula

Theorem

$$\int u(x)\underbrace{v'(x)dx}_{dv} = u(x)v(x) - \int v(x)\underbrace{u'(x)dx}_{du}.$$

In short:

$$\int udv = uv - \int vdu$$

Note The integrand is a product of 2 functions: one of which we choose it to be $u(x)$ and the other to be $v'(x)$. Usually we choose the function which we know its antiderivative as $v'(x)$.

Example

Example

Evaluate $\displaystyle\int x \cos x dx$.

Example

Evaluate $\int x^2 \ln x \; dx$.

Example

Example

Evaluate $\int (t+1)\, e^t \, dt$.

Example

Example

Evaluate $\displaystyle\int \tan^{-1} x \; dx$

Example

Let $I_n = \int x^n e^x dx$, where n is a non-negative integer. Prove that for $n \geq 1$,

$$I_n = x^n e^x - nI_{n-1}.$$

The formula

$$I_n = x^n e^x - nI_{n-1},$$

expresses I_n in terms of I_{n-1}, and $n - 1 < n$. This is known as a reduction formula for $I_n = \int x^n e^x dx$.

Reduction Formula

Example

$I_n = \int x^n e^x dx$, prove that $I_n = x^n e^x - nI_{n-1}$.

Solution

For $n \geq 1$, we use integration by parts, with

$$u(x) = x^n, \ v'(x) = e^x, \ \textit{so that}$$
$$u'(x) = nx^{n-1}, v(x) = e^x.$$
$$I_n = \int x^n e^x dx = x^n e^x - \int n(x^{n-1})e^x \, dx$$
$$= x^n e^x - n \underbrace{\int x^{n-1} e^x \, dx}_{I_{n-1}} = x^n e^x - nI_{n-1}.$$

Reduction Formula

Example

Let $I_n = \int x^n e^x \, dx$, where $n \geq 0$. Use the reduction formula

$$I_n = x^n e^x - nI_{n-1},$$

to determine a formula for I_4.

Solution

$$\begin{aligned} I_4 &= x^4 e^x - 4I_3, \\ I_3 &= x^3 e^x - 3I_2, \\ I_2 &= x^2 e^x - 2I_1 \\ I_1 &= xe^x - I_0. \end{aligned}$$

Note that $I_0 = \int x^0 e^x \, dx = e^x + C.$

Solution

Thus, we obtain,

$$I_1 = xe^x - I_0 = xe^x - \left(e^x + C\right),$$

$$I_2 = x^2e^x - 2I_1 = x^2e^x - 2\left(xe^x - e^x - C\right)$$
$$= x^2e^x - 2xe^x + 2e^x + 2C,$$

$$I_3 = x^3e^x - 3I_2 = x^3e^x - 3x^2e^x + 6xe^x - 6e^x - 6C, \text{ and}$$

$$I_4 = x^4e^x - 4I_3 = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + 24C.$$

Reduction Formulae

More generally, we may let I_n be an indefinite integral or definite integral of the form

$$\int x(\ln x)^n \, dx, \int \cos^n x \, dx, \int \tan^n x \, dx,$$

$$\int \sec^n x \, dx, \quad \int_0^8 (x+1)^n e^{2x} \, dx.$$

If we can express I_n in terms of I_m , where $m < n$, the expression obtained is known as a reduction formula for I_n.