MH1810 Math 1 Part 4 Integration

Volume

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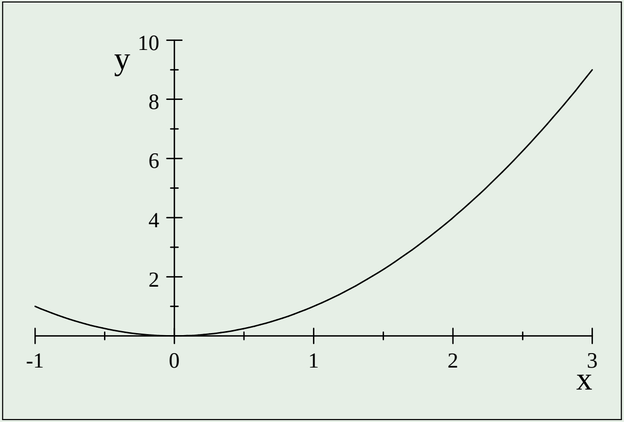
Example

Find the area of the region enclosed by the curve $y = x^2$, $x = 1, x = 3$ and $y = 0$.



Solution

Solution

For $1 \leq x \leq 3$*, the area of a typical "strip" is*

$$x^2 \cdot \delta x.$$

Thus, the area of the bounded region is

$$\lim_{\delta x \to 0} \sum x^2 \cdot \delta x = \int_1^3 x^2 \, dx = \left[\frac{x^3}{3} \right]_1^3 = \frac{26}{3}.$$

Example

Example

Find the area of the region lying above the line $y = 1$ and below the curve $y = \frac{5}{x^2+1}$.

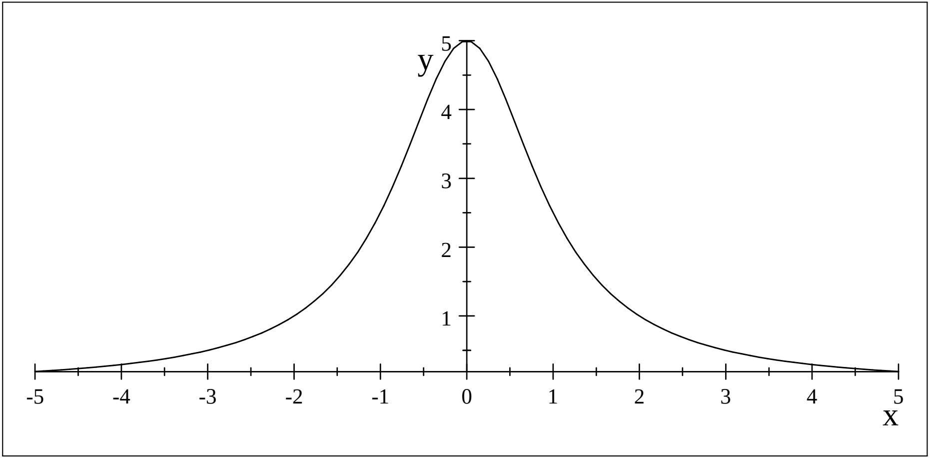
Solution

To find the intersections of $y = 1$ *and* $y = \dfrac{5}{x^2+1}$ *we must solve*

$$1 = \frac{5}{x^2+1},$$

which gives $x^2 + 1 = 5$, *so* $x^2 = 4$ *and* $x = \pm 2$.

Solution



Solution

Solution

For $-2 \leq x \leq 2$*, area of a typical strip is*

$$\left(\frac{5}{x^2+1} - 1\right)(\delta x).$$

Therefore the area of the region is then given by

$$\lim_{\delta x \to 0} \sum \left(\frac{5}{x^2+1} - 1\right) \cdot \delta x = \int_{-2}^{2} \frac{5}{x^2+1} - 1\, dx = \left[(5\tan^{-1} x) - x\right]_{-2}^{2}$$

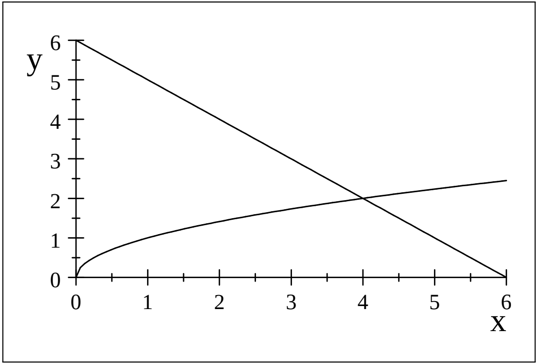
$$= 5(\tan^{-1} 2 - \tan^{-1} -2) - 4$$

$$= 10\tan^{-1} 2 - 4.$$

Example

Example

Evaluate the area of the region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 2$.



Solution

Solution

For $0 \leq y \leq 2$, *note that*

$$y = 6 - x \iff x = 6 - y \text{ and } y = \sqrt{x} \iff x = y^2.$$

The area of a typical horizontal strip is given by

$$\left((6-y) - y^2\right)(\delta y).$$

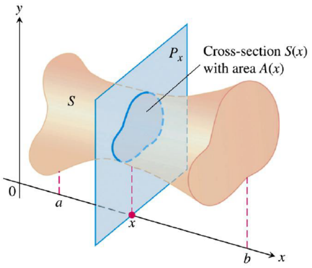
Therefore, the area of the bounded region is given by

$$\lim_{\delta y \to 0} \sum \left((6-y) - y^2\right)(\delta y) = \int_0^2 \left((6-y) - y^2\right)\, dy = \left[6y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2$$

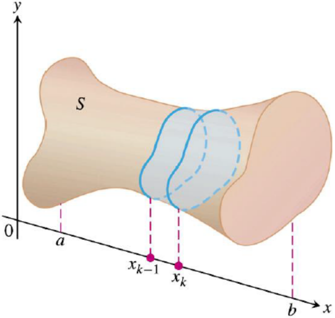
$$= 12 - 2 - \frac{8}{3} = \frac{22}{3}.$$

Volumes Using Cross-Sections

Recall that a cross-section of a solid S is the plane region obtained by intersecting S with a plane.



Suppose a coordinate system is introduced to describe the solid S such that all x coordinates of points in S are in the interval $[a, b]$. At each $x \in [a, b]$, let $A(x)$ denote the cross-section of the solid S. Assume that $A(x)$ is a continuous function.



The volume of each typical slice at x with thickness δx is given by

$$A(x)\,\delta x.$$

Therefore the total volume of the solid is given by

$$V = \lim_{\delta x \to 0} \sum A(x)\,\delta x = \int_a^b A(x)\,dx.$$

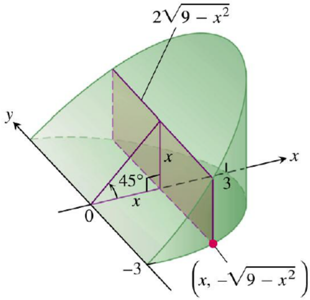
Calculating Volumes Using Cross-Sections

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$ the area of a typical cross-section.
3. Find the limits of integration, i.e., the interval $[a, b]$.
4. Integrate $A(x)$ to find the volume.

Example

Example

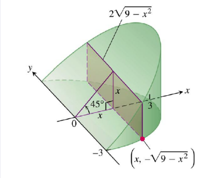
A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° at the centre of the cylinder. Find the volume of the wedge.



Solution

Solution

As typical cross section is a rectangle with width $2\sqrt{9-x^2}$ *and height* x, $0 \le x \le 3$.



The cross sectional area is given by

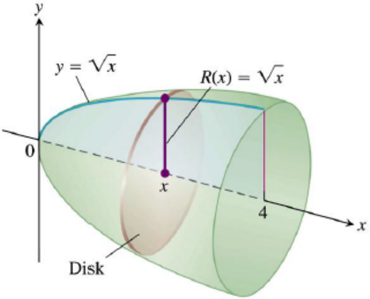
$$A(x) = 2x\sqrt{9-x^2}$$

Thus the volume required is

$$V = \int_0^3 2x\sqrt{9-x^2}dx$$

$$= \left[-\frac{2}{3}\left(9-x^2\right)^{3/2}\right]_0^3 = 18.$$

Solids of revolution are solids obtained by revolving a region about a line.

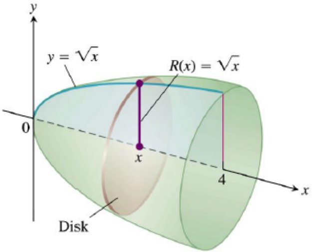


The Disc Method

If we take cross section along the axis of rotation, a typical cross section is a disc.

Example

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$, $0 \le x \le 4$.



Solution

Solution

For $0 \leq x \leq 4$, *the volume of a typical disc is*

$$\pi(\sqrt{x})^2\delta x = \pi x\delta x.$$

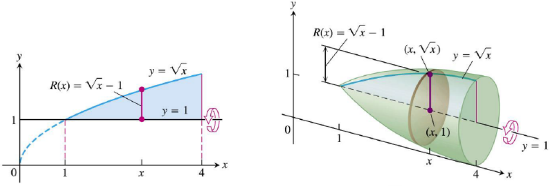
Thus, the volume of the solid is

$$\int_0^4 \pi x \, dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi.$$

Example

Example

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.



(a) (b)

Solution

Solution

For each x, $1 \leq x \leq 4$, the volume of the typical disc is

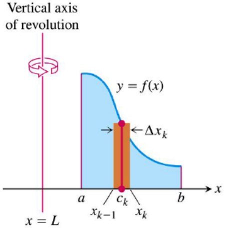
$$\pi(\sqrt{x}-1)^2(\delta x).$$

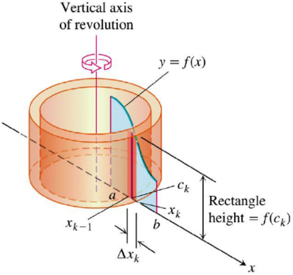
Therefore, the volume of the solid is

$$\begin{aligned} V &= \int_1^4 \pi(\sqrt{x}-1)^2 \, dx \\ &= \pi \int_1^4 \left(x - 2\sqrt{x} + 1\right) dx \\ &= \pi \left[\frac{x^2}{2} - \frac{4}{3}x^{3/2} + x\right]_1^4 = \frac{7}{6}\pi. \end{aligned}$$

The Cylindrical Shell Method

We note that : Volume of a cylindrical shell with radius r, height h and thickness t is approximated by $(2\pi r)ht$.





The solid generated by revolution may be interpreted as the "sum" of cylindrical shells.

The Cylindrical Shell Method

Theorem

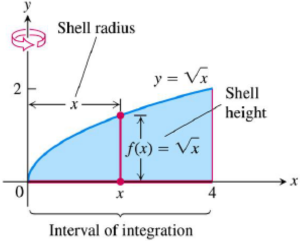
The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0$, $a \leq x \leq b$ and a vertical line is

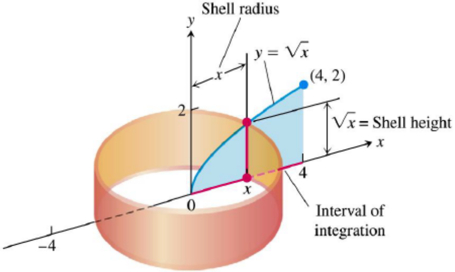
$$V = 2\pi \int_a^b (\textit{shell radius})(\textit{shell height})\, dx.$$

Example

Example

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.





Solution

Solution

A typical shell has height $\sqrt{x}$ *and radius* x, *for* $0 \leq x \leq 4$.
Volume of a typical shell $= 2\pi x \left(\sqrt{x}\right) \delta x$. *Therefore the required volume is*

$$V = 2\pi \int_0^4 x^{3/2} dx = \frac{128}{5}\pi$$