Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 11**

Topics: Reduction Formulae, Improper Integrals, Areas, Volumes.

1. Let $I_n = \int \cos^n x \, dx$ for $n = 0, 1, 2, 3, \cdots$.

   (a) Prove the reduction formula

   $$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \text{ for } n \geq 2.$$

   (b) Use part(a) to evaluate

   (i) $\int \cos^3 x \; dx$.

   (ii) $\int_0^{\pi/2} \cos^4 x \; dx$.

[Solution]

(a) For $n \geq 2$, we have

$$I_n = \int \cos^n x \, dx = \int \underbrace{\left(\cos^{n-1} x\right)}_{u} \underbrace{\cos x}_{v'} \; dx$$

$$= \underbrace{\left(\cos^{n-1} x\right)}_{u} \underbrace{(\sin x)}_{v} - \int \underbrace{(n-1)\left(\cos^{n-2} x\right)(-\sin x)}_{u'} \underbrace{\sin x}_{v} \; dx$$

$$= \left(\cos^{n-1} x\right)(\sin x) + \int (n-1)\left(\cos^{n-2} x\right)\left(\sin^2 x\right) \; dx$$

$$= \left(\cos^{n-1} x\right)(\sin x) + \int (n-1)\left(\cos^{n-2} x\right)\left(1 - \cos^2 x\right) \; dx$$

$$= \left(\cos^{n-1} x\right)(\sin x) + (n-1) \underbrace{\int \left(\left(\cos^{n-2} x\right) dx\right.}_{I_{n-2}} - (n-1) \underbrace{\int \left(\left(\cos^{n-2} x\right) \cos^2 x\right) \; dx}_{I_n}.$$

Therefore, we have

$$I_n = \left(\cos^{n-1} x\right)(\sin x) + (n-1) I_{n-2} - (n-1) I_n$$

which gives

$$n I_n = \left(\cos^{n-1} x\right)(\sin x) + (n-1) I_{n-2},$$

and hence

$$I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$$

(b) Use part(a) to evaluate

(i) $\int \cos^3 x \; dx = I_3 = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} I_1$

$= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \int \cos x \; dx = \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C$.

(ii) $\displaystyle\int \cos^4 x\ dx = I_4 = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4} I_2$

$\displaystyle = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2} I_0\right)$ where $I_0 = \int 1\ dx = x + C_1$.

Thus, we have

$$\int_0^{\pi/2} \cos^4 x\ dx = \left[\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left(\frac{1}{2}\cos x \sin x + \frac{1}{2}x\right)\right]_0^{\pi/2} = \frac{3\pi}{16}$$

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2. Evaluate each of the following improper integrals.

(a) $\displaystyle\int_0^1 x \ln x\ dx$

(b) $\displaystyle\int_0^1 \frac{4r}{\sqrt{1-r^4}} dr$

(c) $\displaystyle\int_1^\infty \frac{1}{x^3}\ dx$

(d) $\displaystyle\int_5^\infty \frac{1}{\sqrt{x-1}}\ dx$

(e) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\ dx$

[Solution]

(a) $\displaystyle\int_0^1 x \ln x\ dx$ is improper since $\ln 0$ is undefined. We have

$$\int_0^1 x \ln x\ dx = \lim_{a\to 0^+} \int_a^1 x \ln x\ dx.$$

Using integration by parts, we have

$$\int_a^1 x \ln x\ dx = \left[\frac{x^2}{2}(\ln x)\right]_a^1 - \int_a^1 \frac{x^2}{2}\,\frac{1}{x}\ dx = \frac{-a^2}{2}(\ln a) - \left[\frac{x^2}{4}\right]_a^1 = \frac{-a^2}{2}(\ln a) - \frac{1}{4} + \frac{a^2}{4}$$

Therefore, we have

$$\int_0^1 x \ln x\ dx = \lim_{a\to 0^+}\left(\frac{-a^2}{2}(\ln a) - \frac{1}{4} + \frac{a^2}{4}\right).$$

We evaluate the first limit as follows:

$$\underbrace{\lim_{a\to 0^+}\left(\frac{a^2}{2}(\ln a)\right)}_{indet\ `0\cdot\infty'} = \underbrace{\lim_{a\to 0^+}\left(\frac{\ln a}{2/a^2}\right)}_{`\frac{\infty}{\infty}'} \overset{L'H}{=} \lim_{a\to 0^+}\left(\frac{\frac{1}{a}}{-4/a^3}\right) = \lim_{a\to 0^+}\left(\frac{a^2}{-4}\right) = 0.$$

Thus, we have $\displaystyle\int_0^1 x \ln x\ dx = -\frac{1}{4}$.

The improper integral converges to $-\dfrac{1}{4}$.

(b) $\displaystyle\int_0^1 \frac{4r}{\sqrt{1-r^4}}dr = \underbrace{\lim_{b\to 1^-}\left(\int_0^b \frac{4r}{\sqrt{1-r^4}}dr\right) = \lim_{b\to 1^-}\left(2\int_0^{b^2} \frac{1}{\sqrt{1-u^2}}du\right)}_{\text{subst. } u=r^2} = \lim_{b\to 1^-} 2\left[\sin^{-1}(u)\right]_0^{b^2}$

$\displaystyle = \lim_{b\to 1^-} 2\left(\sin^{-1}(b^2)\right) = 2(\frac{\pi}{2}) = \pi$

The improper integral converges to π.

(c) $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left[\frac{1}{-2x^2}\right]_1^b = \lim_{b\to\infty}\left(\frac{1}{-2b^2}+\frac{1}{2}\right) = \frac{1}{2}$

The improper integral converges to $\dfrac{1}{2}$.

(d) $\displaystyle\int_5^\infty \frac{1}{\sqrt{x-1}}\,dx = \lim_{b\to\infty}\left(\int_5^b \frac{1}{\sqrt{x-1}}\,dx\right) = \lim_{b\to\infty}\left[2\sqrt{x-1}\right]_5^b = \lim_{b\to\infty}\left(2\sqrt{b-1}-4\right) = \infty$

The improper integral diverges.

(e) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty}\left(\int_2^b \frac{1}{x(\ln x)^2}\,dx\right) = \lim_{b\to\infty}\left[\frac{-1}{\ln x}\right]_2^b = \lim_{b\to\infty}\left(\frac{-1}{\ln b}+\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$

The improper integral converges to $\dfrac{1}{\ln 2}$.

. . . .

3. Sketch each of the region enclosed by the given lines and curves. Find the area of the enclosed region.

(a) $y = 2x - x^2$ and $y = -3$

(b) $y = x^2 - 2x$ and $y = x$

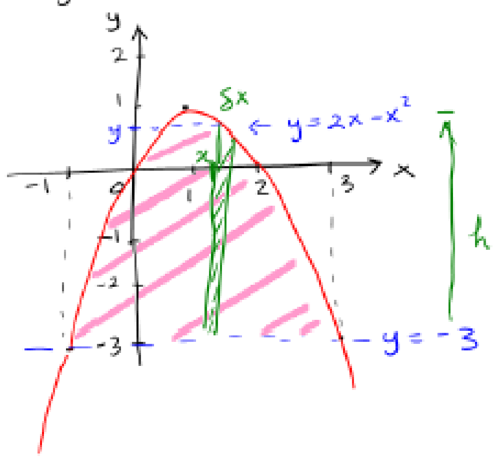
(c) $x = y^2$ and $x = y + 2$

(d) $x = y^3 - y^2$ and $x = 2y$

(Answers: (a) $\frac{32}{3}$ (b) $\frac{9}{2}$ (c) $\frac{9}{2}$ (d) $\frac{37}{12}$)

[Solution] For the above, sketch the graphs to find the enclosed regions.

(a) $y = 2x - x^2$, $y = -3$



For $-1 \leq x \leq 3$,

area of a typical strip is

$$\underbrace{(2x - x^2 - (-3))}_{h}(\delta x)$$

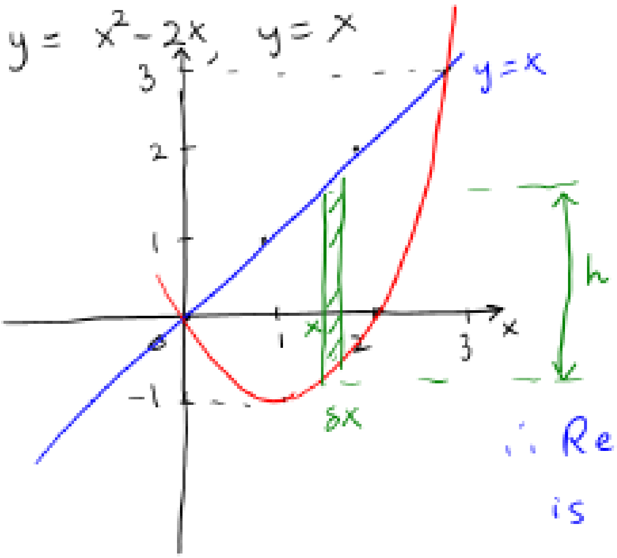
So, area of region is

$$\int_{-1}^{3} 2x - x^2 + 3 \, dx$$

$$= \left[x^2 - \frac{x^3}{3} + 3x \right]_{-1}^{3}$$

$$= \frac{32}{3}$$

(b) $y = x^2 - 2x, \quad y = x$



For each $0 \leq x \leq 3$ area of typical strip is

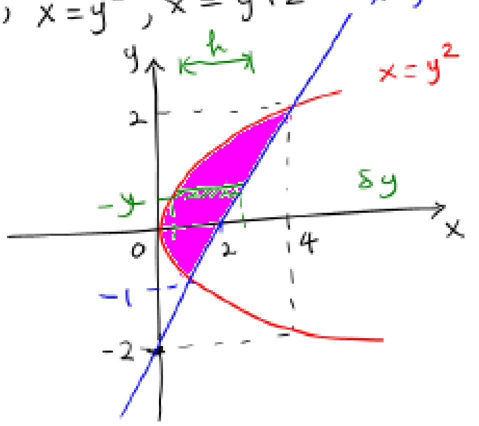
$$\underbrace{(x - (x^2 - 2x))}_{h}(\delta x)$$

$\therefore$ Required area is $\displaystyle\int_0^3 x - (x^2 - 2x)\, dx$

$$= \int_0^3 x^2 - 3x\, dx = \left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_0^3$$

$$= \frac{9}{2} \quad //$$

(c) $x = y^2$, $x = y + 2$



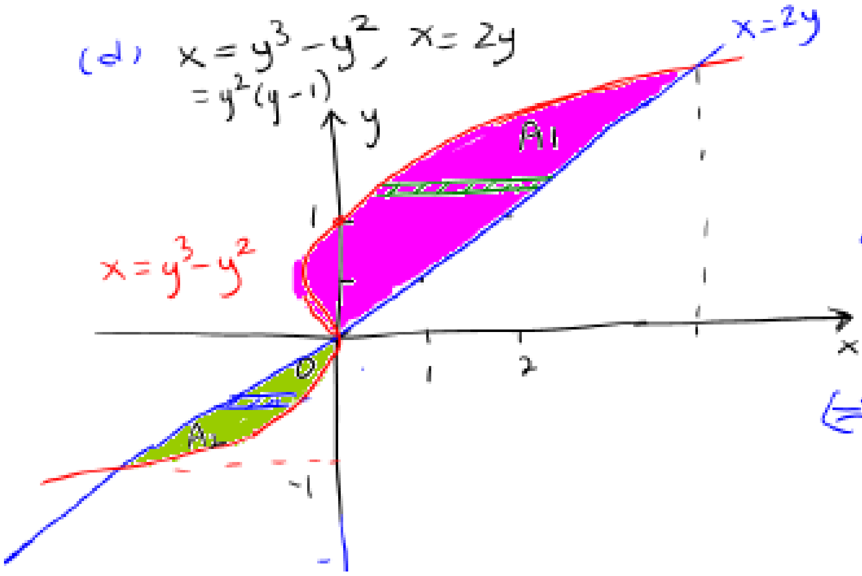
We use horizontal strips:-

For each $-1 \le y \le 2$, area of a typical strip is
$\underbrace{((y+2) - y^2)}_{h}(\delta y)$

So, required area of region is $\int_{-1}^{2} (y+2) - y^2 \, dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^{2}$

$= (2 + 2 - \frac{8}{3}) - (\frac{1}{2} + 2 + \frac{1}{3}) = \frac{9}{2}$

(d) $x = y^3 - y^2$, $x = 2y$
$= y^2(y-1)$



Solve for point of intersection:

$y^3 - y^2 = 2y$

$\Leftrightarrow y^3 - y^2 - 2y = 0$

$\Leftrightarrow y(y^2 - y - 2) = 0$

$\Leftrightarrow y(y+1)(y-2) = 0$

$\Leftrightarrow y = 0$ or $y = -1$ or $y = 2$

(d):

For region A_1, note that $0 \le y \le 2$, the area of a typical horizontal strip is $(2y - (y^3 - y^2))(\delta y)$.
Thus, the area of the region A_1 is

$$\int_0^2 (2y - (y^3 - y^2)) \, dy = \left[y^2 - \frac{y^4}{4} + \frac{y^3}{3} \right]_0^2 = \frac{8}{3}$$

For region A_2, note that $-1 \le y \le 0$, the area of a typical horizontal strip is $((y^3 - y^2) - 2y)(\delta y)$.

Thus, the area of the region A_1 is

$$\int_{-1}^{0} ((y^3 - y^2) - 2y)\ dy = \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2\right]_{-1}^{0} = -(1/4) - 1/3 + 1 = \frac{5}{12}$$

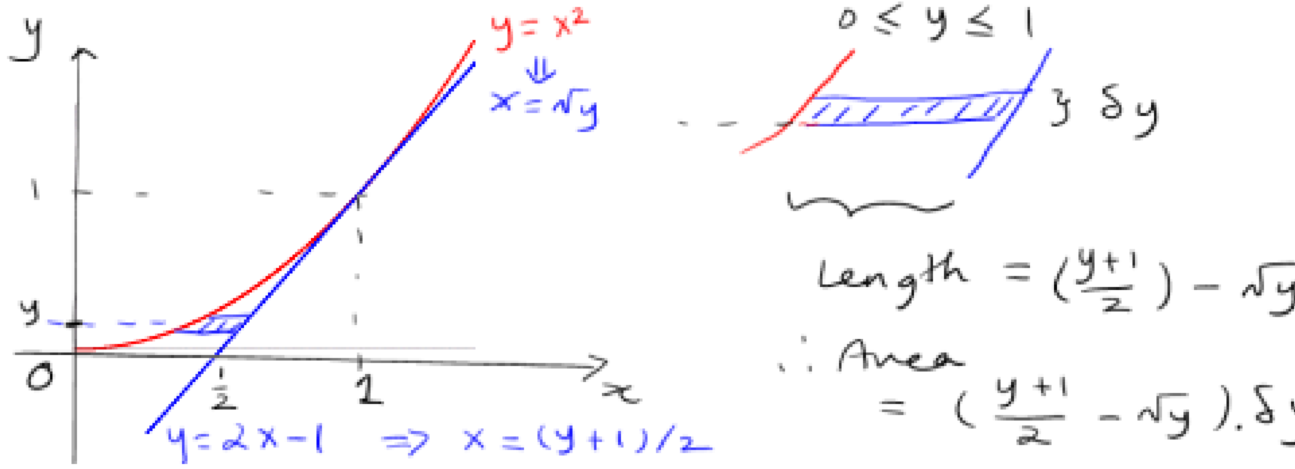
Total area is $\frac{8}{3} + \frac{5}{12} = \frac{37}{12}$.

....

(a) Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

[Solution] For $y = x^2$, we have $\frac{dy}{dx} = 2x$. Thus the equation of tangent line to this parabola at $(1, 1)$ is $y - 1 = 2(x - 1)$, i.e., $y = 2x - 1$.

From the sketch of the graph of $y = x^2$ and $y = 2x - 1$,



Note: $y = 2x - 1 \Leftrightarrow x = \frac{y+1}{2}$.

At each y where $0 \leq y \leq 1$, the area of each horizontal strip is $(\frac{y+1}{2} - \sqrt{y}) \cdot \delta y$.

Therefore, the area required is

$$\int_0^1 \left[\frac{1}{2}(y+1) - \sqrt{y}\right] \, dy = \frac{1}{4}y^2 + \frac{1}{2}y - \frac{2}{3}(\sqrt{y})^3|_0^1 = \frac{1}{12}.$$

Alternative solution:

The area required is the difference between the area of the region under the curve $y = x^2$ for $0 \leq x \leq 1$ and the area of the region (which is a triangle) under the line $y = 2x - 1$, for $1/2 \leq x \leq 1$.

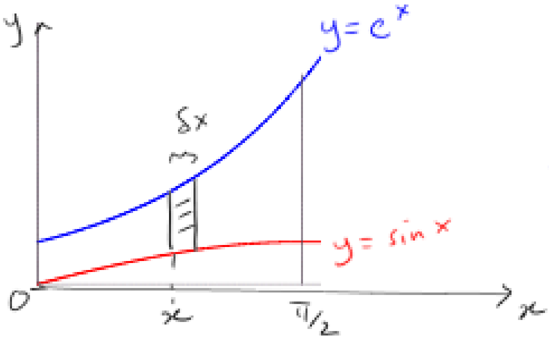
Thus we have area required is

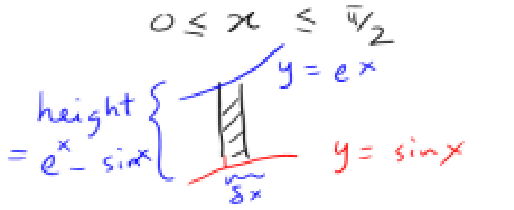
$$\int_0^1 x^2 dx - \frac{1}{2}(1)(\frac{1}{2}) = \left[\frac{x^3}{3}\right]_0^1 - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

(b) Sketch the region bounded by the given curves and find the area of the region:

$$y = \sin x, y = e^x, x = 0, x = \pi/2$$

[Solution] You should sketch the graphs of $y = \sin x, y = e^x$ on $[0, \pi/2]$:





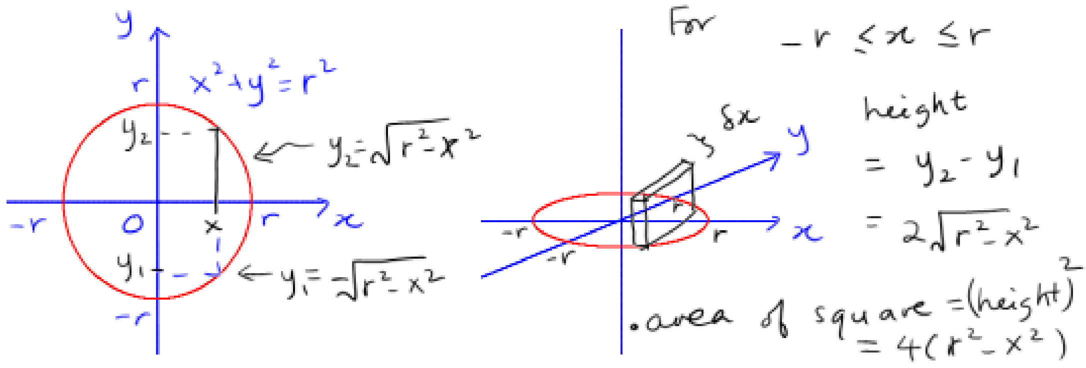
Note that for each $0 \leq x \leq \frac{\pi}{2}$, the area of the vertical strip is $(e^x - \sin x)\delta x$.
The area of the region is thus given by

$$\int_0^{\frac{\pi}{2}} e^x - \sin x \; dx = (e^x + \cos x)\Big|_0^{\frac{\pi}{2}} = e^{\frac{\pi}{2}} - 2.$$

4. The base of a solid S is circular disk with radius r. Parallel cross-sections perpendicular to the base are squares. Show that the volume of the solid S is $\dfrac{16}{3}r^3$.

(Note that the equation of a circle with radius r and centered at $(0,0)$ is $x^2+y^2=r^2$.)

[Solution] Consider the circular disc with equation $x^2+y^2=r^2$, centered at $(0,0)$ with radius r.



For each x where $-r \le x \le r$, note that the area of each square cross-section is

$$A(x) = (2y)^2 = (2\sqrt{r2 - x^2})^2 = 4(r^2 - x^2).$$

Thus, for $-r \le x \le r$, the volume of a square slice is $4(r^2 - x^2)\delta x$.

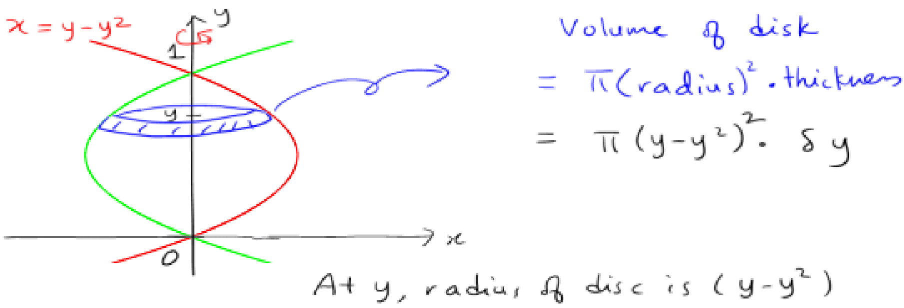
So, the volume of solid is

$$\int_{-r}^{r} 4(r^2 - x^2)\ dx = \int_{-r}^{r} 4(r^2 - x^2)\ dx = 4\left[r^2x - \frac{1}{3}x^3\right]_{-r}^{r} = \frac{16}{3}r^3$$

. . . .

5. (a) Find the volume of the solid obtained by revolving the region bounded by the curves $x = y - y^2$ and $x = 0$ about the y-axis. (Answer: $\frac{\pi}{30}$)

[Solution] Sketch a diagram and note that:



For each y with $0 \le y \le 1$: a cross section is a disk with radius $y - y^2$ and thickness δy so that its volume is $\pi(y - y^2)^2(\delta y)$.

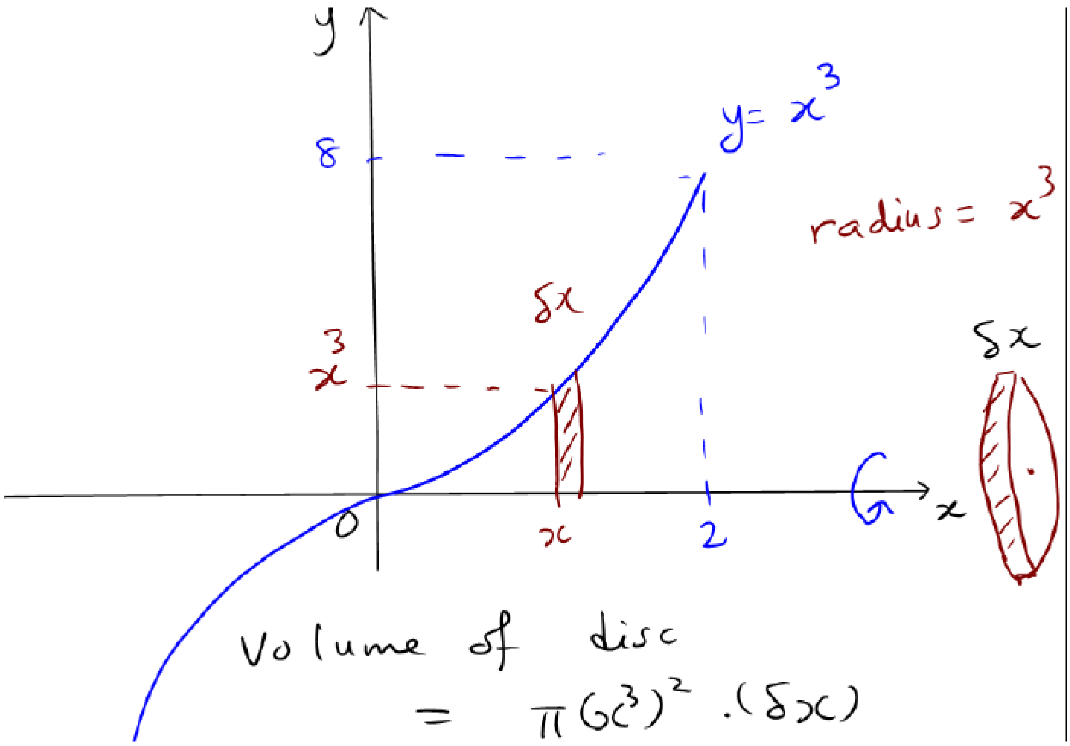
Thus, the volume of the solid is $\displaystyle\int_0^1 \pi(y - y^2)^2\ dy = \pi \int_0^1 y^2 - 2y^3 + y^4\ dy$

$$= \pi\left(y^3/3 - 2y^4/2 + y^5/5\right)\Big|_0^1 = \frac{\pi}{30}.$$

. . . .

(b) Find the volume of the solid generated by revolving the regions bounded by the curve $y = x^3$ and lines $y = 0$ and $x = 2$ about the x-axis.

[Solution]



Using the disc method, we have the volume of the solid is given by

$$\int_0^2 \pi(x^3)^2\, dx = \left[\pi \frac{x^7}{7}\right]_0^2 = \frac{128\pi}{7}.$$

Method using Cylindrical shell: We shall use a horizontal slice with $0 \le y \le 8$. This horizontal slice generates a cylindrical shell when it is rotated about the x-axis.

Volume of a typical cylindrical shell is $2\pi(y)(2 - y^{1/3})(\delta y)$.

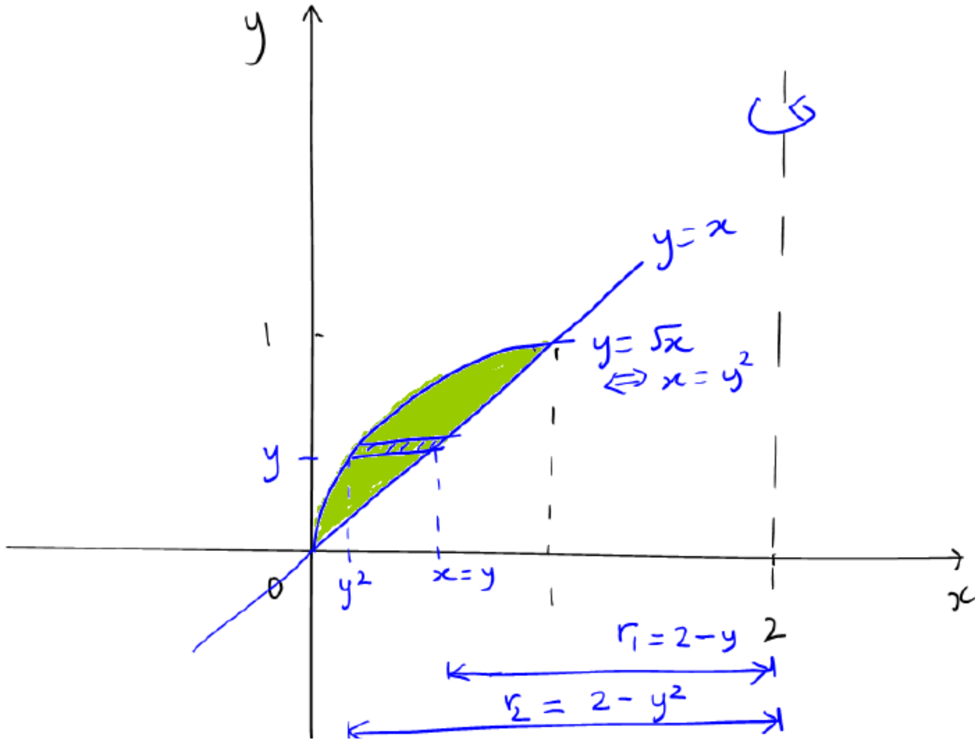
Thus the volume of the solid is

$$\int_0^8 2\pi(y)(2 - y^{1/3})dy = 2\pi \int_0^8 \left(2y - y^{5/3}\right) dy = 2\pi \left[y^2 - \frac{3y^{7/3}}{7}\right]_0^8 = \frac{128\pi}{7}.$$

. . . .

(c) Find the volume of the solid obtained by revolving the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 2$.(Answer: $\frac{8}{15}\pi$)

[Solution] Sketch the graph and note that



At each y between 0 and 1:

each horizontal slice is ring (center from the line $x = 2$): with thickness δy, with outer radius $2 - y^2$ and inner radius is $2 - y$.

Thus the volume of typical slice is

$$\pi\left((2 - y^2)^2 - (2 - y)^2\right)\delta y.$$

Therefore volume of the solid is

$$\int_0^1 \pi\left[(2 - y^2)^2 - (2 - y)^2\right]\, dy = \pi\int_0^1 \left[(4 - 4y^2 + y^4) - (4 - 4y + y^2)\right]\, dy$$

$$= \pi\int_0^1 \left(-5y^2 + y^4 + 4y\right)\, dy = \pi\left[\frac{-5y^3}{3} + \frac{y^5}{5} + 2y^2\right]_0^1 = \left(\frac{-5}{3} + \frac{1}{5} + 2\right)\pi = \frac{8}{15}\pi.$$

Method by Cylindrical shell:

For each x where $0 \le x \le 1$, each vertical strip rotated about the line $x = 2$ will generate a cylindrical shell with radius $(2 - x)$ and height $(\sqrt{x} - x)$. Thus, the volume of a typical cylindrical shell is $2\pi(2 - x)(\sqrt{x} - x)(\delta x)$.

Therefore, the volume of the solid is

$$\int_0^1 2\pi(2 - x)(\sqrt{x} - x)\, dx = 2\pi\int_0^1 (2 - x)(\sqrt{x} - x)\, dx$$

$$= 2\pi \int_0^1 \left(2\sqrt{x} - 2x - x^{3/2} + x^2\right)\ dx = 2\pi \left[\frac{4x^{3/2}}{3} - x^2 - \frac{2x^{5/3}}{5} + \frac{x^3}{3}\right]_0^1$$

$$= 2pi \left(\frac{4}{3} - 1 - \frac{2}{5} + \frac{1}{3}\right) = \frac{8}{15}\pi.$$

. . . .

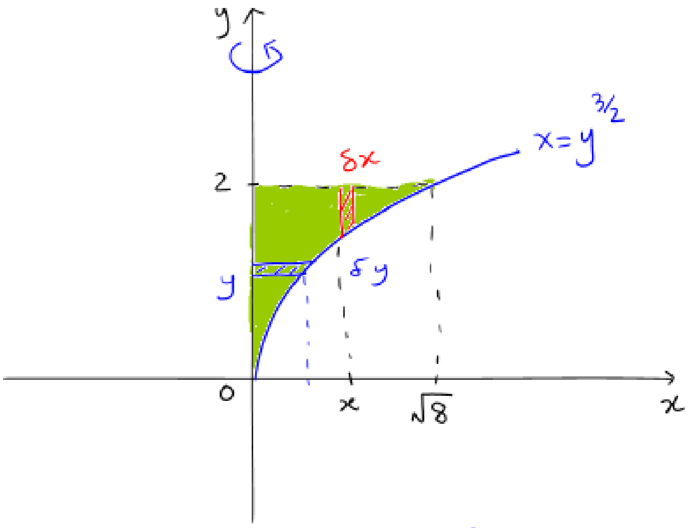
6. Find the volume of the solid generated by revolving the regions bounded by the lines and curves about the y-axis.

(a) $x = y^{3/2}$, $x = 0$, $y = 2$

(b) $x = \sqrt{2 \sin 2y}$, $0 \leq y \leq \pi/2$, $x = 0$

[Solution]

(a) Volume of the solid is $\int_0^2 (\pi y^3)\, dy = \pi \left[\frac{y^4}{4}\right]_0^2 = 4\pi$



① Disc method

Volume of solid $= \int_0^2 \pi (y^{3/2})^2\, dy$

$= \pi \int_0^2 y^3 dy = \pi \left(\frac{y^4}{4}\right)\Big|_0^2$

$= 4\pi.$

② Cylindrical shell

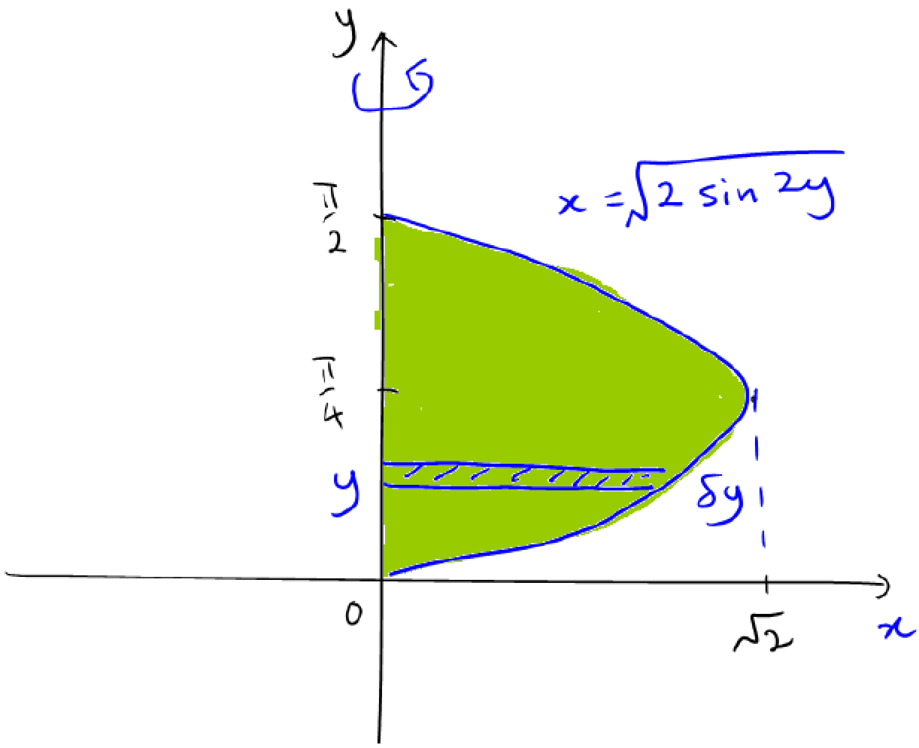
Volume of solid $= \int_0^{\sqrt{8}} 2\pi(x)(2 - x^{2/3})\, dx$

$= 2\pi \int_0^{\sqrt{8}} 2x - x^{5/3}\, dx$

$= 2\pi \left[x^2 - \frac{x^{8/3}}{8/3} \right]_0^{\sqrt{8}}$

$= 2\pi \left(8 - \frac{16(3)}{8}\right) = 4\pi.$

(b) Volume of the solid is $\int_0^{\pi/2} \pi(\sqrt{2\sin(2y)})^2 \ dy = \pi \int_0^{\pi/2} 2\sin(2y) \ dy = \pi \left[-\cos(2y)\right]_0^{\pi/2} = 2\pi.$



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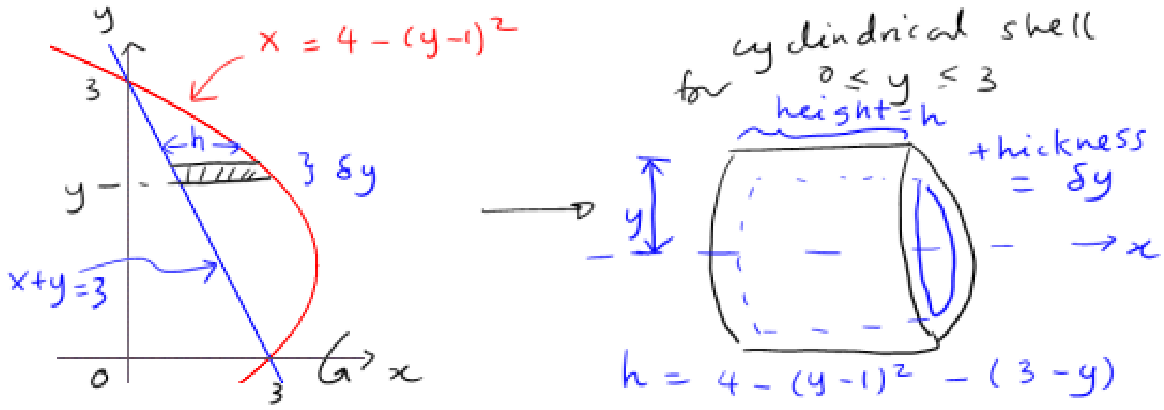
7. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves $x + y = 3$ and $x = 4 - (y-1)^2$ about the x-axis.

[Solution] For $0 \le y \le 3$,



The volume δV of a typical cylindrical shell is

$$\delta V = 2\pi y \underbrace{\left[4 - (y-1)^2 - (3-y)\right]}_{\text{Height of cylindrical shell}} \delta y$$

Thus, the volume V of the solid is

$$\int_0^3 2\pi y[4 - (y-1)^2 - (3-y)]\, dy = 2\pi \int_0^3 y(-y^2 + 3y)\, dy$$

$$= 2\pi \int_0^3 -y^3 + 3y^2\, dy = 2\pi \left[\frac{-y^4}{4} + y^3\right]_0^3 = \frac{27\pi}{2}$$

. . . .

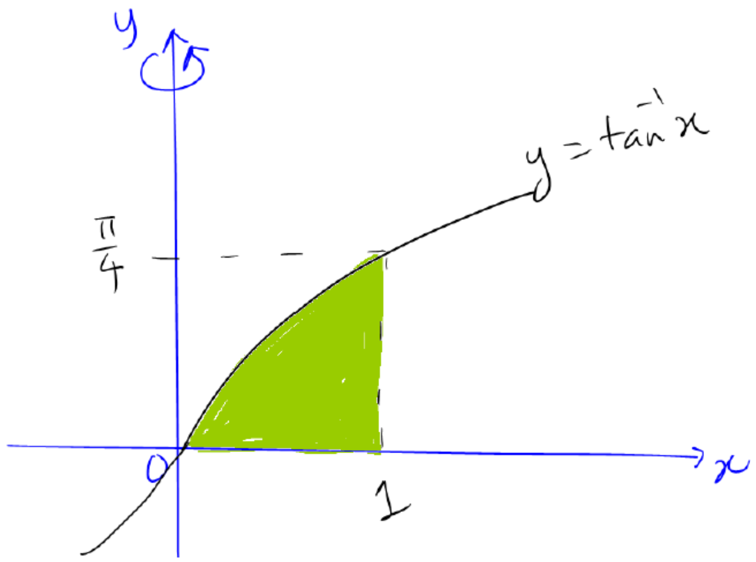
8. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$ and $x = 1$.

   (a) Find the area of the region.

   (b) Find the volume of the solid formed by revolving this region about the y-axis.

[Solution]

(a) Note that for $0 \le x \le 1$ and $0 \le y \le \pi/4$, we have $y = \tan^{-1} x \iff x = \tan y$.



Area of the region is $\displaystyle\int_0^1 \tan^{-1} x \, dx = \left[x \tan^{-1}\right]_0^1 - \int_0^1 x \left(\frac{1}{1+x^2}\right) \, dx$

$$= \frac{\pi}{4} - (\frac{1}{2}) \left[\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

Alternative Method: Area of the enclosed region is

$$\int_0^{\pi/4} 1 - \tan y \, dy = [y - \ln|\sec x| \,]_0^{\pi/4} = \frac{\pi}{4} - (\ln\sqrt{2} - \ln 1) = \frac{\pi}{4} - \ln\sqrt{2} = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

(b) Volume of each cylindrical shell is $(2\pi x)(\tan^{-1} x)\delta x$ where $0 \le x \le 1$.

Thus the volume of the solid is $\displaystyle\int_0^1 (2\pi x)(\tan^{-1} x) \, dx = 2\pi \int_0^1 x \tan^{-1} x \, dx$.

Now, we apply integration by parts to $\displaystyle\int_0^1 x \tan^{-1} x \, dx$:

$u = \tan^{-1} x, v' = x$ so that $u' = \frac{1}{1+x^2}$ and $v = \frac{x^2}{2}$

$$\int_0^1 \underbrace{x}_{v'} \underbrace{\tan^{-1} x}_{u} \, dx = \left[\frac{x^2}{2} \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x^2}{2} (\frac{1}{1+x^2}) \, dx$$

$$= \frac{\pi}{8} - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2}\right) \, dx = \frac{\pi}{8} - \frac{1}{2} \left[x - \tan^{-1}(x)\right]_0^1 = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}.$$

Thus, the volume is $2\pi(\frac{\pi}{4} - \frac{1}{2}) = \pi(\frac{\pi}{2} - 1)$.

Alternative Method: The volume of the solid is

$$\int_0^{\pi/4} \left(\pi(1)^2 - \pi(\tan y)^2\right)\ dy = \pi \int_0^{\pi/4} \left(1 - \tan^2 y\right)\ dy$$

$$= \pi \int_0^{\pi/4} \left(1 - (\sec^2 y - 1)\right)\ dy = \pi\left[2y - \tan y\right]_0^{\pi/4} = \pi(\frac{\pi}{2} - 1)$$