Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 2**

Reading: Lecture slides on Vectors & Thomas' Calculus: Chapter 12 (Sections 12.1 -12.5).

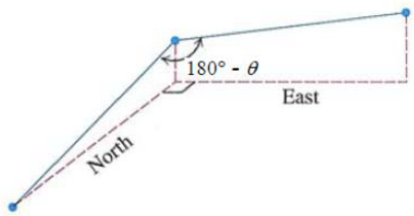
1. Let $\mathbf{u} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.
   Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$, $\mathbf{v} \times \mathbf{u}$, and $\text{proj}_{\mathbf{v}}\mathbf{u}$.

2. For which values of k are $\mathbf{x} = (k, k, 1)$ and $\mathbf{y} = (k, 5, 6)$ in $\mathbb{R}^3$ perpendicular to each other (i.e., orthogonal)? (Answers: $k = -2$ or $k = -3$.)

3. Consider the parallelogram $ABPC$ with adjacent sides AB and AC and vertices $A\,(1, 0, 0)$, $B\,(0, 1, 0)$ and $C\,(0, 0, 1)$.
   (a) Find the area of the parallelogram.
   (b) Find the coordinates of the vertex P.
   (c) Find the acute angle between the diagonals of the parallelogram.

4. Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) along the line from the origin to the point $(1, 1)$. (Distance measured in metres).

5. A water main is to be constructed with at 20% grade (i.e., slope $= \dfrac{\text{height}}{\text{horizontal distance}} = 0.2$) in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east (i.e., the angle θ you need to bend the water main).



6. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$.
   (a) Using $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ and some properties of dot products, prove that
   $$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \pm 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2.$$
   Hence prove that
   $$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2.$$
   (b) Use part (a) to prove that two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$. Also, interpret this geometrically in $\mathbb{R}^2$.

7. (a) Find the vector equation of the line through $A\,(1, 0, 1)$ and $B\,(1, -1, 1)$.
   (b) Find the parametric equation of the line through $P\,(1, 2, -1)$ and $Q\,(-1, 0, 1)$.

(c) Find the parametric equation of the line through the point $R(2,4,5)$ and perpendicular to the plane $3x+7y-5z=21$.

8. Consider vectors $\mathbf{u}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{v}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

   (a) Find a unit vector that is perpendicular to vectors $\mathbf{u}$ and $\mathbf{v}$.

   (b) Determine the scalar equation of the plane Π which passes through the point $(1,1,0)$ and is parallel to $\mathbf{u}$ and $\mathbf{v}$. What is the distance between planes Π and the plane containing the origin and parallel to $\mathbf{u}$ and $\mathbf{v}$?

9. (a) Find the vector equation and scalar equation of the plane through the point $P(1,-1,3)$ parallel to the plane $3x+y+z=7$.

   (b) Find the vector equation of the plane through $A(1,-2,1)$ perpendicular to OA.

10. (a) Find the distance from $S(3,-1,4)$ to the line $\ell: x=4-t, y=3+2t, z=-5+3t$.

    (b) Find the distance from $S(2,-3,4)$ to the plane $x+2y+2z=13$.

    (c) Find the distance between the two planes $x+2y+6z=1$ and $x+2y+6z=10$.

11. Consider four distinct points $A(0,0,0)$, $B(1,2,0)$, $C(0,-3,2)$ and $D(3,-4,5)$ where AB, AC and AD are three edges of a parallelepiped.

    (a) Find the volume of the parallelepiped via scalar triple product.

    (b) If A,B and C are three vertices on the base of the parallelepiped, compute the height of the parallelepiped.

    (c) Let ℓ_1 be the line through A and B and ℓ_2 the line through D and parallel to AC. What is the distance between the skew lines ℓ_1 and ℓ_2?

Challenging Problem

1. **(a)** Prove the Cauchy-Schwarz Inequality:

$$x_1y_1+x_2y_2+x_3y_3 \le \sqrt{x_1^2+x_2^2+x_3^2}\sqrt{y_1^2+y_2^2+y_3^2}$$

   for all $x_i, y_i \in \mathbb{R}$.

   (b) Let a_1,a_2,a_3,b_1,b_2,b_3 be positive real numbers such that $a_1+a_2+a_3=b_1+b_2+b_3$. Prove that

$$\frac{a_1^2}{a_1+b_1}+\frac{a_2^2}{a_2+b_2}+\frac{a_3^2}{a_3+b_3} \ge \frac{a_1+a_2+a_3}{2}.$$

ANSWERS

1. $\sqrt{27}$; $\sqrt{6}$; 8; $4\mathbf{i}-9\mathbf{j}-\mathbf{k}$; $-4\mathbf{i}+9\mathbf{j}+\mathbf{k}$; $\frac{8}{6}(2\mathbf{i}+\mathbf{j}-\mathbf{k})=\frac{8}{3}\mathbf{i}+\frac{4}{3}\mathbf{j}-\frac{4}{3}\mathbf{k}$

2. $k=-2$ or $k=-3$

   (a) The area is $\sqrt{3}$.

   (b) $(-1,1,1)$

   (c) $\pi/2$

3. $5J$.

4. $\theta = \arccos(\frac{2}{\sqrt{104}\sqrt{101}})$ or $\theta \approx 1.55$ rad, or $88.88°$.

7. (a) $\mathbf{r} = (1, 0, 1) + t(0, -1, 0), t \in \mathbb{R}$.

   (b) $x = 1 - 2t, y = 2 - 2t, z = -1 + 2t, t \in \mathbb{R}$.

   (c) $x = 2 + 3t, y = 4 + 7t, z = 5 - 5t, t \in \mathbb{R}$.

8. (a) $\frac{1}{\sqrt{53}}(-4, 6, -1)$

   (b) $-4x + 6y - z = 2$, and the distance is $\dfrac{2}{\sqrt{53}}$.

9. (a) Vector equation: $\mathbf{r} \cdot (3, 1, 1) = 5$, Scalar equation: $3x + y + z = 5$

   (b) Vector equation: $\mathbf{r} \cdot (1, -2, 1) = 6$.

10. (a) $\frac{9\sqrt{42}}{7}$, (b) 3, (c) $\frac{9}{\sqrt{41}}$

11. (a) 5, (b), Height of the parallelepiped $= \frac{5}{\sqrt{29}}$, (c) $\frac{5}{\sqrt{29}}$.