Nanyang Technological University
SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 2**

Reading: Lecture slides on Vectors & Thomas' Calculus: Chapter 12 (Sections 12.1 -12.5).

1. Let $\mathbf{u} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.
   Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$, $\mathbf{v} \times \mathbf{u}$, and $\text{proj}_{\mathbf{v}}\mathbf{u}$.

   Note:
$$\text{proj}_{\mathbf{v}}\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

   (Answers: $\sqrt{27}$; $\sqrt{6}$; 8; $4\mathbf{i} - 9\mathbf{j} - \mathbf{k}$; $-4\mathbf{i} + 9\mathbf{j} + \mathbf{k}$; $\frac{8}{6}(2\mathbf{i} + \mathbf{j} - \mathbf{k}) = \frac{8}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{4}{3}\mathbf{k}$.)

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2. For which values of k are $\mathbf{x} = (k, k, 1)$ and $\mathbf{y} = (k, 5, 6)$ in $\mathbb{R}^3$ perpendicular to each other (i.e., orthogonal)? (Answers: $k = -2$ or $k = -3$.)

   [Solution] $\mathbf{x} = (k, k, 1)$ and $\mathbf{y} = (k, 5, 6)$ in $\mathbb{R}^3$ are perpendicular to each other (i.e., orthogonal) if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

   Thus we have $\mathbf{x} \cdot \mathbf{y} = (k, k, 1) \cdot (k, 5, 6) = k^2 + 5k + 6 = 0$ which gives $k = -2$ or $k = -3$.

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3. Consider the parallelogram $ABPC$ with adjacent sides AB and AC and vertices $A\,(1, 0, 0)$, $B\,(0, 1, 0)$ and $C\,(0, 0, 1)$.

   (a) Find the area of the parallelogram.
   (b) Find the coordinates of the vertex P.
   (c) Find the acute angle between the diagonals of the parallelogram.

   [Solution]

   (a) The area of the parallelogram is $\|\overrightarrow{AB} \times \overrightarrow{AC}\|$. where
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$
$$= (1, 1, 1)$$

   Thus, the area is $\sqrt{3}$.

   (b) To find the coordinates of the vertex P, we note that $\overrightarrow{BP} = \overrightarrow{AC}$, i.e., $\overrightarrow{0P} - \overrightarrow{0B} = \overrightarrow{OC} - \overrightarrow{0A}$.
   Thus, we have $\overrightarrow{0P} = \overrightarrow{0B} + \overrightarrow{OC} - \overrightarrow{0A} = (-1, 1, 1)$.

   (c) The acute angle between the diagonals of the parallelogram is the acute angle between vectors $\overrightarrow{AP}$ and $\overrightarrow{BC}$, where $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (-2, 1, 1)$ and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (0, -1, 1)$.
   We shall use dot-product to find this angle.
   Since $\overrightarrow{AP} \cdot \overrightarrow{BC} = 0$, the angle is a right angle.

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4. Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude $5\ N$) along the line from the origin to the point $(1, 1)$. (Distance measured in metres).

   [Solution] Recall: Work done is $\mathbf{F} \cdot \mathbf{d}$.

   Thus, the work done is $5\mathbf{i} \cdot (\mathbf{i} + \mathbf{j}) = 5J$.

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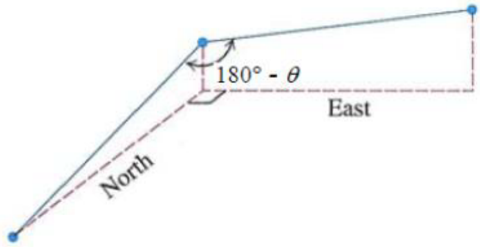
5. A water main is to be constructed with at 20% grade (i.e., slope $= \frac{\text{height}}{\text{horizontal distance}} = 0.2$) in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east (i.e., the angle θ you need to bend the water main).



[SOLUTION] Set the positive x-axis along the east, the positive y-axis along the north, and the positive z-axis along the upward direction.

Let $\mathbf{u} = 10\mathbf{j} + 2\mathbf{k}$ which is parallel to the pipe in the north direction and $\mathbf{v} = 10\mathbf{i} + \mathbf{k}$ which is parallel to the pipe in the east direction.

To find the angle θ , we use dot-product of $\mathbf{u}$ and $\mathbf{v}$.

The angle θ required satisfies

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \ \|\mathbf{u}\| \ \cos\theta;$$

which is $2 = \sqrt{104}\sqrt{101}\cos\theta$.

Thus, $\cos\theta = \frac{2}{\sqrt{104}\sqrt{101}}$ and hence $\theta \approx 1.55$ rad, or $88.88°$.

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6. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$.

(a) Using $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ and some properties of dot products, prove that

$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \pm 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2.$$

Hence prove that

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2.$$

(b) Use part (a) to prove that two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$. Also, interpret this geometrically in $\mathbb{R}^2$.

[SOLUTION]

(a) We use $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ and properties

$$(\mathbf{u} \pm \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) \pm (\mathbf{v} \cdot \mathbf{w}),$$

$$\mathbf{w} \cdot (\mathbf{u} \pm \mathbf{v}) = (\mathbf{w} \cdot \mathbf{u}) \pm (\mathbf{w} \cdot \mathbf{v}), \text{ and}$$

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v}.$$

$$\begin{aligned}\|\mathbf{u} \pm \mathbf{v}\|^2 &= (\mathbf{u} \pm \mathbf{v}) \cdot (\mathbf{u} \pm \mathbf{v}) = ((\mathbf{u} \pm \mathbf{v}) \cdot \mathbf{u}) \pm ((\mathbf{u} \pm \mathbf{v}) \cdot \mathbf{v}) \\ &= ((\mathbf{u} \cdot \mathbf{u} \pm \mathbf{v} \cdot \mathbf{u}) \pm ((\mathbf{u} \cdot \mathbf{v} \pm \mathbf{v} \cdot \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} \pm 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 \pm 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2.\end{aligned}$$

$$\begin{aligned}&\frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2 = \frac{1}{4}\left((\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})\right) - \frac{1}{4}\left((\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})\right) \\ &= \frac{1}{4}\left(\mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}\right) - \frac{1}{4}\left(\mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}\right) = \frac{1}{4}\left(4\mathbf{u} \cdot \mathbf{v}\right) = \mathbf{u} \cdot \mathbf{v}.\end{aligned}$$

WARNING: The following are incorrect:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\underbrace{\|\mathbf{u}\|\ \|\mathbf{v}\|}_{WRONG!} + \|\mathbf{v}\|^2.$$

$$\|\mathbf{u} + \mathbf{v}\|^2 = \underbrace{\mathbf{u}^2}_{WRONG!} + 2\mathbf{u} \cdot \mathbf{v} + \underbrace{\mathbf{v}^2}_{WRONG!}.$$

(b) Note that the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$ is a rectangle when $\mathbf{u}$ and $\mathbf{v}$ are perpendicular. Thus, the result says that a parallelogram is a rectangle if and only if its two diagonals have the same length.

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7. (a) Find the vector equation of the line through $A\,(1,0,1)$ and $B\,(1,-1,1)$.
   (b) Find the parametric equation of the line through $P\,(1,2,-1)$ and $Q\,(-1,0,1)$.
   (c) Find the parametric equation of the line through the point $R\,(2,4,5)$ and perpendicular to the plane $3x+7y-5z=21$.

   [Solution] To obtain an equation of a line, we must find a direction vector and the position vector of a point on this line.

   (a) Positive vector of a point on this line $\overrightarrow{OA}=(1,0,1)$
   Direction vector : $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=(0,-1,0)$.
   Thus, a vector equation of this line is : $\mathbf{r}=(1,0,1)+t(0,-1,0), t\in\mathbb{R}$.
   (b) Positive vector of a point on this line $\overrightarrow{OP}=(1,2,-1)$
   Direction vector $\mathbf{v}=(-2,-2,2)$.
   Parametric equation: $x=1-2t, y=2-2t, z=-1+2t, t\in\mathbb{R}$.
   (c) Direction vector is the normal to the plane. Thus, we have $\mathbf{v}=(3,7,-5)$.
   Parametric equation: $x=2+3t, y=4+7t, z=5-5t, t\in\mathbb{R}$.

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8. Consider vectors $\mathbf{u}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{v}=2\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

   (i) Find a unit vector that is perpendicular to vectors $\mathbf{u}$ and $\mathbf{v}$.
   (ii) Determine the scalar equation of the plane Π which passes through the point $(1,1,0)$ and is parallel to $\mathbf{u}$ and $\mathbf{v}$. What is the distance between planes Π and the plane containing the origin and parallel to $\mathbf{u}$ and $\mathbf{v}$?

   [SOLUTION] NOTE: A plane is determined by a point on this plane and a normal vector (usually denoted by $\mathbf{n}$). Since the plane is parallel to $\mathbf{u}$ and $\mathbf{v}$, the vector $\mathbf{n}=\mathbf{u}\times\mathbf{v}$ is a normal vector.

   (i) Let $\mathbf{n}=\mathbf{u}\times\mathbf{v}=\left(\begin{vmatrix}1 & 2\\ 1 & -2\end{vmatrix}, -\begin{vmatrix}1 & 2\\ 2 & -2\end{vmatrix}, \begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}\right)=(-4,6,-1)$. Thus, a unit vector required is $\hat{\mathbf{n}}=\frac{\mathbf{n}}{\|\mathbf{n}\|}=\frac{1}{\sqrt{53}}(-4,6,-1)$.
   (ii) The scalar equation of the plane Π which passes through $(1,1,0)$ and is parallel to $\mathbf{u}$ and $\mathbf{v}$ is $-4x+6y-z=2$.
   Note that the point O lies on the plane containing both $\mathbf{u}$ and $\mathbf{v}$.
   The distance between planes Π and the plane containing $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\left|\overrightarrow{OP}\cdot\hat{\mathbf{n}}\right|=\left|(1,1,0)\cdot\frac{1}{\sqrt{53}}(-4,6,-1)\right|=\frac{2}{\sqrt{53}}.$$

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9. (a) Find the vector equation and scalar equation of the plane through the point $P\,(1,-1,3)$ parallel to the plane $3x+y+z=7$.
   (b) Find the vector equation of the plane through $A\,(1,-2,1)$ perpendicular to OA.

   [Solution]

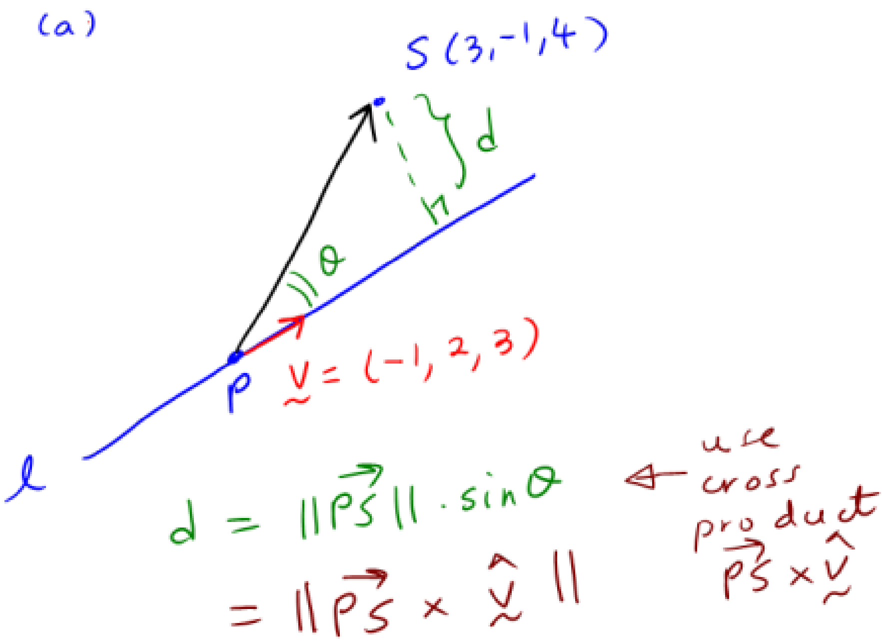
   (a) From the equation of the plane $3x+y+z=7$, we obtain a normal vector $\mathbf{n}=(3,1,1)$ of the plane via the coefficients of x, y and z.
   Thus, a vector equation of this plane is given by $\overrightarrow{PR}\cdot\mathbf{n}=0$, i.e., $(\mathbf{r}-(1,-1,3))\cdot(3,1,1)=0$; which gives $\mathbf{r}\cdot(3,1,1)=5$. Scalar equation: $3x+y+z=5$.
   (b) Normal vector is $\mathbf{n}=\overrightarrow{OA}$. Vector equation : $(\mathbf{r}-(1,-2,1))\cdot(1,-2,1)=0$, i.e., $\mathbf{r}\cdot(1,-2,1)=6$.

10. (a) Find the distance from $S\,(3,-1,4)$ to the line $\ell : x = 4 - t, y = 3 + 2t, z = -5 + 3t$.
(b) Find the distance from $S\,(2,-3,4)$ to the plane $x + 2y + 2z = 13$.
(c) Find the distance between the two planes $x + 2y + 6z = 1$ and $x + 2y + 6z = 10$.

[Answers] (a) $\dfrac{9\sqrt{42}}{7}$ (b) 3 (c) $\dfrac{9}{\sqrt{41}}$.

[Solution] It is helpful to draw a diagram to have an idea of the problem, and use your diagram to find the distance via dot-product or cross-product.

(a)



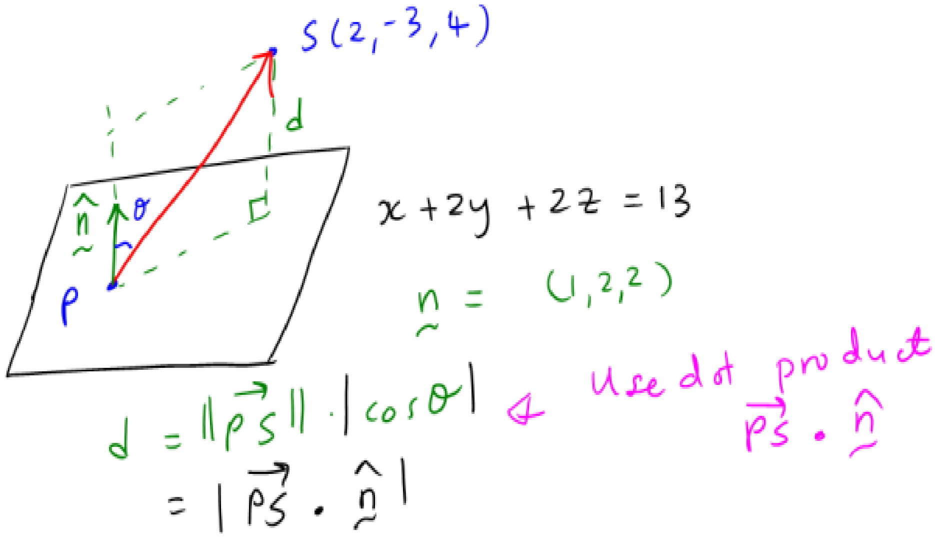
To find the distance from $S\,(3,-1,4)$ to the line $\ell : x = 4 - t, y = 3 + 2t, z = -5 + 3t$, we need a point P on the line. We take $\overrightarrow{OP} = (4, 3, -5)$, with $t = 0$.
A direction vector is t $\mathbf{v} = (-1, 2, 3)$.
The required distance is given by

$$d = \left\|\overrightarrow{PS}\right\| \sin\theta = \left\|\overrightarrow{PS} \times \hat{\mathbf{v}}\right\| = \left\|(-1,-4,9) \times \frac{1}{\sqrt{14}}(-1,2,3)\right\|$$

$$= \frac{1}{\sqrt{14}}\left\|(-30,-6,-6)\right\| = \frac{6}{\sqrt{14}}\left\|(5,1,1)\right\| = \frac{9\sqrt{42}}{7}.$$

(b)

(b)

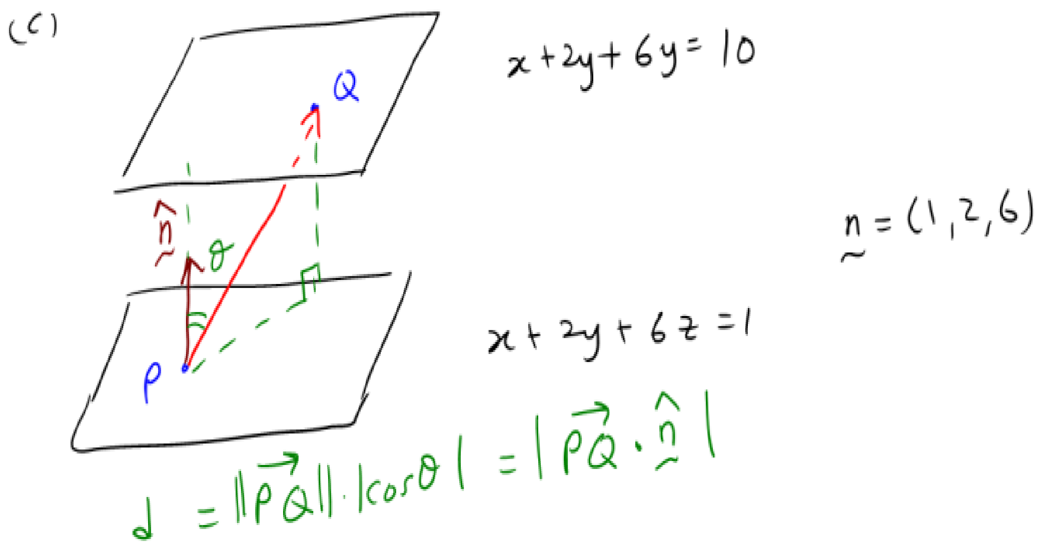


To find the distance from $S\,(2, -3, 4)$ to the plane $x + 2y + 2z = 13$, we begin with finding a point P on the plane by setting $y = 0$ and $z = 0$ to get $x = 13$. Thus $\overrightarrow{OP} = (13, 0, 0)$, and a normal vector $\mathbf{n} = (1, 2, 2)$.
The required distance is given by

$$d = \left\|\overrightarrow{PS}\right\| |\cos\theta| = \left|\overrightarrow{PS} \cdot \widehat{\mathbf{n}}\right|$$

$$= \left|(-11, -3, 4) \cdot \frac{1}{\sqrt{9}}(1, 2, 2)\right| = \left|\frac{-11 - 6 + 8}{3}\right| = 3.$$

(c)



To find the distance between the two planes $x + 2y + 6z = 1$ and $x + 2y + 6z = 10$, we find two points P and Q on each plane as follows:
Set $y = 0$ and $z = 0$ in both equations gives $x = 1$ and $x = 10$ respectively. Thus, we have $\overrightarrow{OP} = (1, 0, 0)$ and $\overrightarrow{OQ} = (10, 0, 0)$ so that $\overrightarrow{PQ} = (9, 0, 0)$.
A normal vector $\mathbf{n} = (1, 2, 6)$.
The required distance is given by

$$d = \left\|\overrightarrow{PQ}\right\| |\cos\theta| = \left|\overrightarrow{PQ} \cdot \hat{\mathbf{n}}\right|$$
$$= \left|(9, 0, 0) \cdot \frac{1}{\sqrt{41}}(1, 2, 6)\right| = \frac{9}{\sqrt{41}} \text{ or } \frac{9\sqrt{41}}{41}.$$

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11. Consider four distinct points $A\,(0,0,0)$, $B\,(1,2,0)$, $C\,(0,-3,2)$ and $D\,(3,-4,5)$ where AB, AC and AD are three edges of a parallelepiped.

(i) Find the volume of the parallelepiped via scalar triple product.

(ii) If A, B and C are three vertices on the base of the parallelepiped, compute the height of the parallelepiped.

(iii) Let ℓ_1 be the line through A and B and ℓ_2 the line through D and parallel to AC. What is the distance between the skew lines ℓ_1 and ℓ_2?

[SOLUTION]

(i) $\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j}$, $\overrightarrow{AC} = -3\mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{AD} = 3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$.

Volume is $\left|\left(\overrightarrow{AB} \times \overrightarrow{AC}\right) \cdot \overrightarrow{AD}\right|$ where $\overrightarrow{AB} \times \overrightarrow{AC} = (4, -2, -3)$.

Thus the required volume is

$$|(4, -2, -3) \cdot (3, -4, 5)| = 5.$$

(ii) Volume of parallelepiped = base area x height.

Base area of parallelepiped $= \left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \|(4, -2, -3)\| = \sqrt{29}$.

Height of the parallelepiped $= \frac{5}{\sqrt{29}}$.

(iii) Note that lines lie on opposite planes, one of which contains the parallelogram with sides AB and AC.

Thus the distance between ℓ_1 and ℓ_2 is height of the parallelepiped which $\frac{5}{\sqrt{29}}$.

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