Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 3**

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1. Find matrices $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ whose sizes and entries satisfy the stated conditions.

   (a) A is 3×4 and $a_{ij} = \begin{cases} 1 & \text{if} \quad i < j, \\ 7 & \text{if} \quad i = j, \\ 9 & \text{if} \quad i > j. \end{cases}$

   (b) B is 4×4 and $b_{ij} = \begin{cases} 1 & \text{if} \quad |i-j| > 1, \\ -1 & \text{if} \quad |i-j| \leq 1. \end{cases}$

   (c) C is 3×2 and $C_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 2^i 3^j & \text{if} \quad i \neq j. \end{cases}$

   (Answers: $A = \begin{bmatrix} 7 & 1 & 1 & 1 \\ 9 & 7 & 1 & 1 \\ 9 & 9 & 7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 18 \\ 12 & 1 \\ 24 & 72 \end{bmatrix}$.)

2. Consider the matrices

   $$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

   Compute the following (where possible). Note M^T means the transpose of M. It is obtained from M by interchanging the rows and columns. Example: For $M = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, we have $M^T = \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$.

   (a) BA (b) BC (c) $D^T - E^T$ (d) $(D-E)^T$ (e) DE (f) $(DA)^T$

   (Answers: (a) Not defined;
   (b) $\begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$; (c) & (d) $\begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$; (e) $\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}$; (f) $\begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$.)

3. Let

   $$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

   (a) Find the third column of AA.
   (b) Find the second row of AB.
   (c) Find the first row of $(AB)^T$.

   (Answers: (a) $\begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$; (b) $(64 \;\; 21 \;\; 59)$; (c) $[67 \;\; 64 \;\; 63]$.)

4. Indicate whether the statement is always true or sometimes false. Justify your answer with a logical statement or a counterexample.

   (a) If A is a square matrix with two identical rows, then AA has two identical rows.

   (b) If A is a square matrix and AA has a column of zeros, then A must have a column of zeros.

   (c) If the matrix sum $AB + BA$ is defined, then A and B must be square.

5. Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

   (a) Verify that $A^2 - 6A + 5I = 0$.

   (b) Use part (a) to find a matrix B such that $AB = I$ and $BA = I$. Explain why A is invertible and find its inverse.
   (Answers: (b) $A^{-1} = (\frac{6}{5}I - \frac{1}{5}A)$).

6. Find A where
   (a) $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$ (b) $(I + 2A)^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$
   (Answers: (a) $A = \frac{1}{7}\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$; (b) $A = \begin{bmatrix} -\frac{5}{14} & -\frac{1}{14} \\ \frac{1}{14} & -\frac{4}{14} \end{bmatrix}$)

7. Consider the matrix A where $A = \begin{bmatrix} 2 & k-2 \\ 3 & k \end{bmatrix}$.

   (a) State all values of k for A to be invertible.

   (b) Suppose A is invertible. Find the inverse A^{-1} and use it to find the solution of the simultaneous equations
   $$\begin{array}{rcrcl} 2x & + & (k-2)y & = & 4 \\ 3x & + & ky & = & 5 \end{array}$$

   (c) State the value of k for which there are no solutions to the simultaneous solutions. Justify your answer.
   (Answers: (a) $k \neq 6$, (b) $A^{-1} = \frac{1}{6-k}\begin{pmatrix} k & -(k-2) \\ -3 & 2 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6-k}\begin{pmatrix} 10-k \\ -2 \end{pmatrix}$ (c) $k = 6$)

8. Suppose A and B are $n \times n$ matrices.

   (a) Prove that $(AB)^T = B^T A^T$.

   (b) Prove that if A is invertible, then A^T is invertible and its inverse is $(A^{-1})^T$.

9. Suppose the first row of an $n \times n$ matrix A is identical to the second row of A. Is there an $n \times n$ matrix B such that $AB = I$? Is A invertible?

10. The trace of a square matrix is defined to be the sum of its diagonal entries. We denote by $\mathrm{tr}(A)$ the trace of a square matrix A.

    (a) Find the trace of each of the following matrices.

    $$X = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 6 & 1 & 3 & 9 \\ -1 & 1 & 2 & 3 \\ 0 & 1 & -7 & 3 \\ 4 & 1 & 3 & 0 \end{bmatrix}$$

    (b) Let $A = [a_{ij}]$. Express $\mathrm{tr}(A)$ in terms of a_{ij}.

(c) Prove: If A and B are $n \times n$ matrices, then $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$.

(d) What can you say about $\text{tr}(\alpha A)$? (You may continue to explore other properties of the trace function. eg, Is $\text{tr}(AB) = \text{tr}(A) \cdot \text{tr}(B)$?)

(Answers: (a) $\text{tr}(X) = 3 + 2 = 5, \text{tr}(Y) = 5, \text{tr}(Z) = 0$.)

11. Campus Yogurt sells three types of yogurt: nonfat, regular, and super creamy at three locations. Location N sells 50 gallons of nonfat, 100 gallons of regular, 50 gallons of super creamy each day. Location C sells 10 gallons of nonfat and Location S sells 60 gallon of nonfat each day. Daily sales of regular yogurt are 90 gallons at Location C and 120 gallons at Location S. At Location C, 50 gallons of super creamy are sold each day, and 40 gallons of super creamy are sold each day at Location S.

    The income per gallon for nonfat, regular, and super creamy is \$ 12, \$ 10, and \$ 15, respectively.

    Use matrix product to find the daily income at each of the three locations. (Answers: $2350, 1770, 2520$.)

12. A new mass transit system has just gone into operation. The transit authority has made studies that predict the percentage of commuters who will change to mass transit or continue driving their automobile. Based on the following information:

		This	year
		Mass transit	Automobile
Next	Mass transit	0.7	0.2
year	Automobile	0.3	0.8

    For example, 30% of commuters taking mass transit this year will change to driving automobile next year.

    Suppose the population of the area remains constant, and that initially 30 percent of the commutes use mass transit and 70 percent use their automobiles.

    What percentage of commuters will be using the mass transit system after 1 year? After 2 years?

    (Answers: 35% and 37.5%)