Nanyang Technological University
SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 3**

1. Find matrices $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ whose sizes and entries satisfy the stated conditions.

   (a) A is 3×4 and $a_{ij} = \begin{cases} 1 & \text{if} \quad i < j, \\ 7 & \text{if} \quad i = j, \\ 9 & \text{if} \quad i > j. \end{cases}$

   (b) B is 4×4 and $b_{ij} = \begin{cases} 1 & \text{if} \quad |i-j| > 1, \\ -1 & \text{if} \quad |i-j| \leq 1. \end{cases}$

   (c) C is 3×2 and $C_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 2^i 3^j & \text{if} \quad i \neq j. \end{cases}$

   (Answers: $A = \begin{bmatrix} 7 & 1 & 1 & 1 \\ 9 & 7 & 1 & 1 \\ 9 & 9 & 7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 18 \\ 12 & 1 \\ 24 & 72 \end{bmatrix}$.)

   ....

2. Consider the matrices

   $$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

   Compute the following (where possible). Note C^T means the transpose of C. It is obtained from C by interchanging the rows and columns. Example: For $C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, we have $C^T = \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$.

   (a) BA (b) BC (c) $D^T - E^T$ (d) $(D-E)^T$ (e) DE (f) $(DA)^T$

   (Answers: (a) Not defined;

   (b) $\begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$; (c) & (d) $\begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$; (e) $\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}$; (f) $\begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$.)

   ....

3. Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

(a) Find the third column of AA.

(b) Find the second row of AB.

(c) Find the first row of $(AB)^T$.

(Answers: (a) $\begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$; (b) $(64 \quad 21 \quad 59)$; (c) $[67 \quad 64 \quad 63]$.)

[SOLUTION]

(a) Find the third column of AA.
Solution Third column of AA = A [Third Column of A]

$$= \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$$

(b) Find the second row of AB.
Solution Second row of AB = [Second Row of A]B

$$= \begin{bmatrix} 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 36+28 & -12+5+28 & 24+15+20 \end{bmatrix} = \begin{bmatrix} 64 & 21 & 59 \end{bmatrix}$$

(c) Find the first row of $(AB)^T$.
Solution First row of $(AB)^T = (\text{ First column of } (AB))^T = (A[\text{ First Column of } B])^T$

$$= \left(\begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 7 \end{bmatrix} \right)^T = \left(\begin{bmatrix} 67 \\ 64 \\ 63 \end{bmatrix} \right)^T = \begin{bmatrix} 67 & 64 & 63 \end{bmatrix}$$

. . . .

4. Indicate whether the statement is always true or sometimes false. Justify your answer with a logical statement or a counterexample.

(a) If A is a square matrix with two identical rows, then AA has two identical rows.
PROOF. The statement is TRUE.
Let A be an $n \times n$ square matrix whose i-row and jth rows are identical, where $i \neq j$. Then we have
The ith-row of AA = [ith row of A]A = [jth row of A]A = jth-row of AA.

(b) If A is a square matrix and AA has a column of zeros, then A must have a column of zeros.
[SOLUTION] The statement is FALSE.
Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) If the matrix sum $AB + BA$ is defined, then A and B must be square.
[SOLUTION] The statement is TRUE.
Suppose A is $m \times n$ and B is $r \times p$. For both AB and BA to be well-defined, we must have $n = r$ (and AB is $m \times p$) and $p = m$ (where BA is $r \times n$). Thus, we have AB is $m \times m$ and BA is $n \times n$. Moreover, for $AB + BA$ to be defined, we must have $m = n$.
In conclusion, both matrices A and B must be square matrices of the same size.

. . . .

5. Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

(a) Verify that $A^2 - 6A + 5I = 0$.

(b) Hence, use the definition of invertible matrix, to explain that A is invertible and find its inverse.

[SOLUTION]

(a) $A^2 - 6A + 5I$

$= \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}^2 - 6\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 12 \\ 12 & 13 \end{pmatrix} - 6\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

(b) (The technique used is quite standard: use the definition of invertibility and inverse, i.e., Can we find a matrix B such $AB = I \& BA = I$.)

From the equation $A^2 - 6A + 5I = 0$, we have :

$$A\left(\frac{6}{5}I - \frac{1}{5}A\right) = I \quad \& \quad \left(\frac{6}{5}I - \frac{1}{5}A\right)A = I.$$

Thus, $B = \dfrac{6}{5}I - \dfrac{1}{5}A$.

It follows from the definition of invertibility that A is invertible and (at the same time we have its inverse) $A^{-1} = \left(\frac{6}{5}I - \frac{1}{5}A\right)$.

Remark From the theory of linear systems of equations and matrix equation, we have the following one-sided inverse result: Suppose A is a square matrix. If there is a matrix B such that $AB = I$, then A is invertible and B is its inverse. Similarly, if B is such that $BA = I$, then A is invertible and B is its inverse.

Note: Some common mistakes.

(i) It is incorrect to write: $I = (\frac{6}{5} - \frac{1}{5}A)A$, as $\frac{6}{5}$ is a number and A is a matrix.

(ii) The following argument is not valid:
Multiply $A^2 - 6A + 5I = 0$ throughout by A^{-1} to get $A - 6I + 5A^{-1} = 0$ so that $A^{-1} = \frac{1}{5}(6I - A)$.

This above argument assumes that A is invertible so that A^{-1} exists and we can do multiplication by A^{-1}. However, we have to establish the invertibility of A before we write A^{-1}.

. . . .

6. Find A where

(a) $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$ (b) $(I + 2A)^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

[SOLUTION] Use the formula for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = (ad - bc)^{-1}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

(a)

$$(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix} \Leftrightarrow 7A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} \Leftrightarrow A = \frac{1}{7}\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$$

(b) $(I + 2A)^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\Leftrightarrow I + 2A = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{bmatrix} \Leftrightarrow 2A = \begin{bmatrix} -\frac{5}{7} & -\frac{1}{7} \\ \frac{1}{7} & -\frac{4}{7} \end{bmatrix} \Leftrightarrow A = \begin{bmatrix} -\frac{5}{14} & -\frac{1}{14} \\ \frac{1}{14} & -\frac{4}{14} \end{bmatrix}$$

. . . .

7. Consider the matrix A where $A = \begin{bmatrix} 2 & k-2 \\ 3 & k \end{bmatrix}$.

(a) State all values of k for A to be invertible.

(b) Suppose A is invertible. Find the solution of the simultaneous equations

$$\begin{array}{ccccc} 2x & + & (k-2)y & = & 4 \\ 3x & + & ky & = & 5 \end{array}$$

(c) State the value of k for which there are no solutions to the simultaneous solutions. Justify your answer.

[SOLUTION]

(a) $\det(A) = \begin{vmatrix} 2 & k-2 \\ 3 & k \end{vmatrix} = 2k - 3(k-2) = 6 - k.$

Therefore, A is invertible if and only if $\det(A) = 6 - k \neq 0$, i.e., $k \neq 6$.

(b) The linear system is equivalent to the matrix equation $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

If A is invertible, then $\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$.

Note that $A^{-1} = \frac{1}{6-k} \begin{pmatrix} k & -(k-2) \\ -3 & 2 \end{pmatrix}$.

Thus, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6-k} \begin{pmatrix} k & -(k-2) \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \frac{1}{6-k} \begin{pmatrix} 4k - 5(k-2) \\ 4(-3) + 5(2) \end{pmatrix} = \frac{1}{6-k} \begin{pmatrix} 10 - k \\ -2 \end{pmatrix}.$$

(c) When $k = 6$, the linear system becomes

$$\begin{array}{ccccc} 2x & + & 4y & = & 4 \\ 3x & + & 6y & = & 5 \end{array}$$

which is equivalent to

$$\begin{array}{ccccc} x & + & 2y & = & 2 \\ x & + & 2y & = & 5/3 \end{array}.$$

There is no solution because the linear system represent two parallel lines which do not intersect.

. . . .

8. Suppose A and B are $n \times n$ matrices.

   (a) Prove that $(AB)^T = B^T A^T$.

   (b) Prove that if A is invertible, then A^T is invertible and its inverse is $(A^{-1})^T$.

   [SOLUTION]

   (a) Note that both $(AB)^T$ and $B^T A^T$ are $n \times n$ matrices. It remains to verify that their (i, j)th entries are equal.
   The (i, j)th entry of $(AB)^T$ is the (j, i)th entry of AB which is

$$\sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} (A^T)_{kj} (B^T)_{ik} = \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj},$$

   which is the (i, j)th entry of $B^T A^T$.
   Thus, we have proved $(AB)^T = B^T A^T$.

   (b) Since A is invertible, its inverse A^{-1} exists and $AA^{-1} = I = A^{-1}A$.
   We shall verify that $A^T (A^{-1})^T = I$ and $(A^{-1})^T A^T = I$. We use the result in part(a).

$$A^T (A^{-1})^T = \left(A^{-1}A\right)^T = I^T = I$$

   and

$$(A^{-1})^T A^T = \left(AA^{-1}\right)^T = I^T = I$$

   By definition A^T is invertible and its inverse is $(A^{-1})^T$.

....

9. Suppose the first row of an $n \times n$ matrix A is identical to the second row of A. Is there an $n \times n$ matrix B such that $AB = I$? Is A invertible?

   [SOLUTION]

   Suppose the first column of an $n \times n$ matrix B is identical to the second column of B.

   First row of the product $(AB) =$ (first row of A) $B =$ (second row of $A)B$

   $=$ second row of (AB)

   However, the identity matrix does not have two identical rows.

   Thus, there is no $n \times n$ matrix B such that $AB = I$. Hence A is not invertible.

....

10. The trace of a square matrix is defined to be the sum of its diagonal entries. We denote by $\mathrm{tr}(A)$ the trace of a square matrix A.

   (a) Find the trace of each of the following matrices.

$$X = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 6 & 1 & 3 & 9 \\ -1 & 1 & 2 & 3 \\ 0 & 1 & -7 & 3 \\ 4 & 1 & 3 & 0 \end{bmatrix}$$

   (b) Let $A = [a_{ij}]$. Express $\mathrm{tr}(A)$ in terms of a_{ij}.

   (c) Prove: If A and B are $n \times n$ matrices, then $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.

   (d) What can you say about $\mathrm{tr}(\alpha A)$? (You may continue to explore other properties of the trace function. eg, Is $\mathrm{tr}(AB) = \mathrm{tr}(A) \cdot \mathrm{tr}(B)$?)

[Solution]

(a) $\mathrm{tr}(X) = 3 + 2 = 5, \mathrm{tr}(Y) = 5, \mathrm{tr}(Z) = 0$

(b) Suppose $A = [a_{ij}]$ is an $n \times n$ matrix. Then

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

(c) (Proof.)
Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are an $n \times n$ matrices. Then

$$\mathrm{tr}(A + B) = \sum_{i=1}^{n} (A + B)_{ii} = \sum_{i=1}^{n} (a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \mathrm{tr}(A) + \mathrm{tr}(B).$$

(d)

$$\mathrm{tr}(\alpha A) = \sum_{i=1}^{n} (\alpha A)_{ii} = \sum_{i=1}^{n} \alpha a_{ii} = \alpha \sum_{i=1}^{n} a_{ii} = \alpha \mathrm{tr}(A)$$

Therefore, we have established that $\mathrm{tr}(\alpha A) = \alpha \mathrm{tr}(A)$.

11. Campus Yogurt sells three types of yogurt: nonfat, regular, and super creamy at three locations. Location N sells 50 gallons of nonfat, 100 gallons of regular, 50 gallons of super creamy each day. Location C sells 10 gallons of nonfat and Location S sells 60 gallon of nonfat each day. Daily sales of regular yogurt are 90 gallons at Location C and 120 gallons at Location S. At Location C, 50 gallons of super creamy are sold each day, and 40 gallons of super creamy are sold each day at Location S.

    The income per gallon for nonfat, regular, and super creamy is \$ 12, \$ 10, and \$ 15, respectively.

    Use matrix product to find the daily income at each of the three locations. (Answers: $2350, 1770, 2520$.)

    [Solution]

	in gallons		
	Non Fat	Regular	Creamy
Location N	50	100	50
Location C	10	90	50
Location S	60	120	40
Income per gallon	12	10	15

The daily income at each three locations are respectively I_N, I_C and I_S which are computed as follows:

$$\begin{pmatrix} 50 & 100 & 50 \\ 10 & 90 & 50 \\ 60 & 120 & 40 \end{pmatrix} \begin{pmatrix} 12 \\ 10 \\ 15 \end{pmatrix} = \begin{pmatrix} 2350 \\ 1770 \\ 2520 \end{pmatrix}$$

12. A new mass transit system has just gone into operation. The transit authority has made studies that predict the percentage of commuters who will change to mass transit or continue driving their automobile. Based on the following information:

		This	year
		Mass transit	Automobile
Next	Mass transit	0.7	0.2
year	Automobile	0.3	0.8

For example, 30% of commuters taking mass transit this year will change to driving automobile next year.

Suppose the population of the area remains constant, and that initially 30 percent of the commutes use mass transit and 70 percent use their automobiles.

What percentage of commuters will be using the mass transit system after 1 year? After 2 years?

(Answers: 35% and 37.5%)

[Solution] $P = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$

$\begin{pmatrix} M_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$,

$$\begin{pmatrix} M_1 \\ A_1 \end{pmatrix} = P \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} = \begin{pmatrix} 0.35 \\ 0.65 \end{pmatrix}$$

$$\begin{pmatrix} M_2 \\ A_2 \end{pmatrix} = P \begin{pmatrix} M_1 \\ A_1 \end{pmatrix} = \begin{pmatrix} 0.375 \\ 0.625 \end{pmatrix}$$

Therefore, 35% and 37.5% of commuters will be using the mass transit system after 1 year and after 2 years respectively.