Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 5**

Reference: Thomas' Calculus: Chapter 2, Section 2.1 - 2.2, 2.4 - 2.6.

1. Suppose that $\lim_{x\to 1} p(x) = 4$, $\lim_{x\to 1} q(x) = \pi$ and $\lim_{x\to 1} r(x) = 3$. Determine each of the following limits and justify each step by indicating the appropriate Limit Law(s).

   (a) $\lim_{x\to 1} [\pi p(x) + q(x) - (qr)(x)]$ (b) $\lim_{x\to 1} \dfrac{p(x)+q(x)}{r(x)}$

   [Solution]

   (a) $\lim_{x\to 1} [\pi p(x) + q(x) - (qr)(x)]$
   $= \lim_{x\to 1} [\pi p(x)] + \lim_{x\to 1} q(x) - \lim_{x\to 1} (qr)(x)$ (Limit laws for $\pm$)
   $= \pi \lim_{x\to 1} p(x) + \pi - \left(\lim_{x\to 1} q(x)\right)\left(\lim_{x\to 1} r(x)\right)$ (Limit laws for product)
   $= \pi(4) + \pi - (\pi)(3) = 2\pi$

   (b) Similarly, we apply Limit law for quotient etc.

$$\lim_{x\to 1} \frac{p(x)+q(x)}{r(x)} = \frac{\lim_{x\to 1}(p(x)+q(x))}{\lim_{x\to 1}(r(x))} = \frac{\lim_{x\to 1} p(x) + \lim_{x\to 1} q(x)}{3} = \frac{4+\pi}{3}$$

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2. Find the limit.

   (a) $\lim_{x\to \pi/2} \cos x = \cos(\pi/2) = 0$

   (b) $\lim_{x\to \infty} 179 = 179$

   (c) $\lim_{x\to 3^-} (x^2 + \pi x + \sqrt{2}) = 9 + 3\pi + \sqrt{2}$

   (d) $\lim_{y\to 3} 4^y = 4^3 = 64$ (e) $\lim_{t\to 125} \sqrt[3]{t} = \sqrt[3]{125} = 5$

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3. Use continuity to determine the following limits.

   (a) $\lim_{x\to 1} \sqrt{\dfrac{x}{1+3x}} = \sqrt{\dfrac{1}{1+3(1)}} = \dfrac{1}{2}$

   (b) $\lim_{x\to 1} \sin(x-1)^2 = \sin(1-1)^2 = 0$

   (c) $\lim_{x\to 1} \tan\left(\dfrac{(2-x^2)\pi}{3}\right) = \tan\left(\dfrac{(2-1^2)\pi}{3}\right) = \tan\left(\dfrac{\pi}{3}\right) = \sqrt{3}$

   (d) $\lim_{x\to 3} \ln|x-2| = \ln|3-2| = \ln(1) = 0$

   (e) $\lim_{x\to \sqrt{2}} \tan^{-1}\left(\dfrac{x^2}{2}\right) = \tan^{-1}\left(\dfrac{2}{2}\right) = \dfrac{\pi}{4}$

   [Answer: (a) $\frac{1}{2}$ (b) 0 (c) $\sqrt{3}$ (d) 0 (e) $\frac{\pi}{4}$

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4. Use appropriate techniques to find the following limits.

(a) $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1} = \lim_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x\to 1} \dfrac{(x^2+x+1)}{(x+1)} = \dfrac{3}{2}$

(b) $\lim_{x\to\sqrt{2}} \dfrac{x^2-2}{x-\sqrt{2}} = \lim_{x\to\sqrt{2}} \dfrac{(x^2-2)(x+\sqrt{2})}{(x-\sqrt{2})(x+\sqrt{2})} = \lim_{x\to\sqrt{2}} \dfrac{(x^2-2)(x+\sqrt{2})}{(x^2-2)} = \lim_{x\to\sqrt{2}}(x+\sqrt{2}) = 2\sqrt{2}$

(c) $\lim_{x\to 0^-} \dfrac{x}{\sqrt{x+1}-1} = \lim_{x\to 0^-} \dfrac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)} = \lim_{x\to 0^-} \dfrac{x(\sqrt{x+1}+1)}{x} = \lim_{x\to 0^-} \left(\sqrt{x+1}+1\right) = 2$

(d) $\lim_{t\to -3} \dfrac{t^2-9}{2t^2+7t+3} = \lim_{t\to -3} \dfrac{(t-3)(t+3)}{(2t+1)(t+3)} = \lim_{t\to -3} \dfrac{(t-3)}{(2t+1)} = \dfrac{6}{5}$

(e) $\lim_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1} \dfrac{1-x}{x(x-1)} = \lim_{x\to 1} \dfrac{-1}{x} = -1$

(f) $\lim_{t\to\frac{\pi}{4}} \dfrac{\cos 2t}{\cos t - \sin t} \lim_{t\to\frac{\pi}{4}} \dfrac{\cos^2 t - \sin^2 t}{\cos t - \sin t} = \lim_{t\to\frac{\pi}{4}} (\cos t + \sin t) = \sqrt{2}$

(g) $\lim_{h\to 0} \dfrac{(2+h)^3-8}{h} = \lim_{h\to 0} \dfrac{(8+12h+6h^2+h^3)-8}{h} = \lim_{h\to 0}(12+6h+h^2) = 12.$

(h) $\lim_{x\to 7^+} \dfrac{\sqrt{x+2}-3}{x-7} = \lim_{x\to 7^+} \dfrac{(x+2)-3^2}{(x-7)(\sqrt{x+2}+3)} == \lim_{x\to 7^+} \dfrac{1}{(\sqrt{x+2}+3)} = \dfrac{1}{6}.$

(i) $\lim_{t\to 0^+} \left(\dfrac{1}{t} - \dfrac{1}{t^2+t}\right) = \lim_{t\to 0^+} \left(\dfrac{(t+1)-1}{t(t+1)}\right) = \lim_{t\to 0^+} \left(\dfrac{1}{t+1}\right) = 1.$

(j) $\lim_{x\to 0} \left(x^4 \cos\dfrac{1}{x}\right)$

[Solution] We use Squeeze Theorem to show that $\lim_{x\to 0} \left(x^4 \cos\dfrac{1}{x}\right)$ exists.

For $x \neq 0$, we have

$$-1 \leq \cos\frac{1}{x} \leq 1.$$

Multiplying throughout by x^4, we have

$$-x^4 \leq x^4 \cos\frac{1}{x} \leq x^4.$$

Since $\lim_{x\to 0}(-x^4) = 0$ and $\lim_{x\to 0}(x^4) = 0$, we conclude that $\lim_{x\to 0} x^4 \cos\dfrac{1}{x} = 0$ by Squeeze Theorem.

Remark We cannot use limit law for product because $\lim_{x\to 0}\left(\cos\dfrac{1}{x}\right)$ does not exist. (The values $\cos\frac{1}{x}$ oscillate between -1 and 1 as x approaches 0.)

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5. Determine whether $\lim_{x\to 2} f(x)$ exists where

$$f(x) = \begin{cases} \dfrac{3x-6}{x^2-4} & \text{if } 0 < x < 2, \\ 0 & \text{if } x = 2, \\ \dfrac{x-2}{\sqrt{3-x}-1} & \text{if } 2 < x < 3. \end{cases}$$

[Solution] We use Equal One-sided Theorem.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \dfrac{3x-6}{x^2-4} = \lim_{x\to 2^-} \dfrac{3(x-2)}{(x-2)(x+2)} = \lim_{x\to 2^-} \dfrac{3}{(x+2)} = \dfrac{3}{4}$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{x-2}{\sqrt{3-x}-1} \dfrac{\sqrt{3-x}+1}{\sqrt{3-x}+1} = \lim_{x\to 2^+} \dfrac{(x-2)(\sqrt{3-x}+1)}{3-x-1}$

$= \lim_{x\to 2^+} \dfrac{(x-2)(\sqrt{3-x}+1)}{-(x-2)} = \lim_{x\to 2^+} -(\sqrt{3-x}+1) = -2$

Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, the limit $\lim_{x\to 2} f(x)$ does not exist.

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6. If the product $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, is it always true that $f(x)$ and $g(x)$ must be continuous at $x = 0$? Give reasons to your answer.

[Solution] No. Consider the following functions

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \geq 0 \end{cases} \quad \text{and } g(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases}$$

Then $h(x) = 0$ for every real number x. Thus, h is continuous at $x = 0$. However, both f and g are not continuous at $x = 0$.

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7. Find real constants c and d that makes g continuous at $x = 4$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4, \\ d & \text{if } x = 4, \\ cx + 20 & \text{if } x > 4. \end{cases}$$

[Solution]

For g to be continuous at $x = 4$, we must have $\lim_{x\to 4} g(x) = g(4)$.

Thus, we must have $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^+} g(x) = g(4)$.

Note that $g(4) = d$, $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} \left(x^2 - c^2\right) = 16 - c^2$, and $\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} (cx + 20) = 4c + 20$.

For continuity of g at 4, we need to have $16 - c^2 = 4c + 20 = d$.

$$16 - c^2 = 4c + 20 \iff c^2 + 4c + 4 = 0$$

$$\iff (c+2)^2 = 0$$

$$\iff (c+2) = 0 \iff c = -2$$

The value for the constant c is -2 and $d = 4c + 20 = 12$ for g to be continuous at $x = 4$.

Remark For $x < 4$, note that $g(x) = x^2 - c^2$ is a polynomial of degree 2 for any value of c. Thus g is continuous on $(-\infty, 4)$. Similarly, for $x > 4$, $g(x) = cx + 20$ is also a polynomial (of degree 1 if $c \neq 0$ and it is constant if $c = 0$). Thus, g is continuous on $(4, \infty)$. So, g is continuous on $(-\infty, 4) \cup (4, \infty)$ for any value of the constant c.

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8. Suppose $3x \leq f(x) \leq x^3 + 2$ for $0 \leq x \leq 2$.

   (a) What is $f(1)$?

   (b) Use Squeeze Theorem to evaluate $\lim_{x\to 1} f(x)$.

   (c) Is f continuous at $x = 1$?

   [Solution]

   (a) For $f(1)$, we substitute $x = 1$ into $3x \leq f(x) \leq x^3 + 2$. We have $3 \leq f(1) \leq 3$. Thus, $f(1) = 3$.

   (b) Note that $f(x)$ satisfies the inequality $3x \leq f(x) \leq x^3 + 2$ for $0 \leq x \leq 2$.
   Moreover, $\lim_{x\to 1} 3x = 3$ and $\lim_{x\to 1} x^2 + 2 = 3$.
   By Squeeze Theorem, we have $\lim_{x\to 1} f(x) = 3$.

   (c) From Parts (a) and (b), we have $\lim_{x\to 1} f(x) = f(1)$, i.e., f is continuous at $x = 1$.

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9. Under certain circumstances a rumor spreads according the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

   where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants.

   (a) Find $\lim_{t\to\infty} p(t)$.

   (b) For $a = 10$, $k = 0.5$ and t being measured in hours, how long will it take for 80% of the population to hear the rumor?

   (Answer: (b) ≈ 7.3778 hours)

   [Solution]

   (a) Since $k > 0$, we have

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1}{1 + ae^{-kt}} = \frac{1}{1 + a \cdot 0} = 1.$$

   (b) For $a = 10$, $k = 0.5$ and t being measured in hours, how long will it take for 80% of the population to hear the rumor?
   Suppose $P(t_0) = 0.8$. This means

$$\frac{1}{1 + 10e^{-(0.5)t_0}} = 0.8.$$

$$\frac{1}{1 + 10e^{-(0.5)t_0}} = 0.8 \Longleftrightarrow 1 + 10e^{-(0.5)t_0} = 1/0.8 = 1.25 \Longleftrightarrow 10e^{-(0.5)t_0} = 0.25$$

$$\Longleftrightarrow e^{-(0.5)t_0} = 0.025 \Longleftrightarrow -(0.5)t_0 = \ln 0.025 \approx -3.6889$$

   Thus, $t_0 \approx 7.3778$.
   At 7.3778 hours, 80% of the population has heard the rumour!! (So, don't spread rumour !!)

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10. Determine the following infinite limits.

(a) $\lim\limits_{x \to 1^-} \dfrac{1}{1-x^2} = \lim\limits_{x \to 1^-} \underbrace{\dfrac{1}{(1-x)}}_{\to +\infty} \underbrace{\dfrac{1}{(1+x)}}_{\to 1/2} = +\infty$

(b) $\lim\limits_{x \to 1^+} \dfrac{x}{1-\sqrt{x}} = \lim\limits_{x \to 1^+} \dfrac{x}{1-\sqrt{x}} = -\infty.$

Note: For $x > 1$, we have $\sqrt{x} > 1$ so that $1 - \sqrt{x} < 0$. Thus, $\dfrac{x}{1-\sqrt{x}} < 0$. This explains why it is a negative infinity.