Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 6**

1. Determine the following limits at infinity.

(a) $\lim_{x\to\infty} \dfrac{3x+5}{x-4}$

(b) $\lim_{x\to\infty} \dfrac{x^3-2x+3}{5-2x^2}$

(c) $\lim_{x\to\infty} \dfrac{x+2}{\sqrt{9x^2+1}}$

(d) $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2+4}}$

(e) $\lim_{x\to\infty} \left(\sqrt{x^4+6x^2}-x^2\right)$

(f) $\lim_{x\to\infty} \dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$

(g) $\lim_{x\to\infty} [\ln(2+x)-\ln(1+x)]$

[Solution]

(a) $\lim_{x\to\infty} \dfrac{3x+5}{x-4} = \lim_{x\to\infty} \dfrac{3+\frac{5}{x}}{1-\frac{4}{x}} = \dfrac{3+0}{1-0} = 3.$

(b) $\lim_{x\to\infty} \dfrac{x^3-2x+3}{5-2x^2}$

Method 1 Dividing by the highest power of x from both numerator and denominator, we have

$$\lim_{x\to\infty} \frac{x^3-2x+3}{5-2x^2} = \lim_{x\to\infty} \frac{1-\frac{2}{x^2}+\frac{3}{x^3}}{\frac{5}{x^3}-\frac{2}{x^2}}$$

Note that since

$$\lim_{x\to\infty} 1-\frac{2}{x^2}+\frac{3}{x^3} = 1 \quad \& \quad \lim_{x\to\infty} \frac{5}{x^3}-\frac{2}{x} = 0$$

which indicates that the required limit is an infinite limit. To determine its sign, we note that

$$\frac{5}{x^3}-\frac{2}{x} = \frac{5-2x^2}{x^3} < 0 \text{ for } x \text{ positively large.}$$

Thus we have

$$\lim_{x\to\infty} \frac{x^3-2x+3}{5-2x^2} = \lim_{x\to\infty} \frac{1-\frac{2}{x^2}+\frac{3}{x^3}}{\frac{5}{x^3}-\frac{2}{x^2}} = -\infty.$$

<u>Method 2</u> Since the numerator has higher power of x, we can do a long division first before we proceed to discuss the limiting behaviour. However, be aware that if the mathematical expression is complicated, it may take too much time and effort to do the long division. This method may not be efficient.

$$\lim_{x\to\infty} \frac{x^3-2x+3}{5-2x^2} = \lim_{x\to\infty} -\frac{1}{2}x + \frac{\frac{1}{2}x+3}{5-2x^2} = \lim_{x\to\infty} -\frac{1}{2}x + \frac{\frac{1}{2x}+\frac{3}{x^2}}{\frac{5}{x^2}-2} = -\infty$$

since

$$\lim_{x\to\infty} -\frac{1}{2}x = -\infty, \lim_{x\to\infty} \frac{\frac{1}{2x}+\frac{3}{x^2}}{\frac{5}{x^2}-2} = 0.$$

<u>Method 3</u> Dividing by the highest power of x from the denominator, we have

$$\lim_{x\to\infty} \frac{x^3-2x+3}{5-2x^2} = \lim_{x\to\infty} \frac{x-\frac{2}{x}+\frac{3}{x^2}}{\frac{5}{x^2}-2} = -\infty$$

since

$$\lim_{x\to\infty} x-\frac{2}{x}+\frac{3}{x^2} = +\infty \quad \& \quad \lim_{x\to\infty} \frac{5}{x^2}-2 = -2.$$

(We have '$\frac{\infty}{-2} = -\infty$'.)

. . . .

(c) In this problem, note that we look at $x > 0$. For $x > 0$, we have $x = \sqrt{x^2} = |x|$.

$$\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x^2}}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3}$$

OR

$$\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \frac{x+2}{|x|\sqrt{9+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x^2}}} = \frac{1+0}{\sqrt{9+0}} = \frac{1}{3}$$

OR

$$\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \sqrt{\frac{(x+2)^2}{9x^2+1}} \text{ (Note: } x+2>0 \text{ so that } (x+2) = \sqrt{(x+2)^2}\text{)}$$

$$\lim_{x\to\infty} \sqrt{\frac{x^2(1+2/x)^2}{x^2(9+1/x^2)}} = \lim_{x\to\infty} \sqrt{\frac{(1+2/x)^2}{9+1/x^2}} = \sqrt{\frac{(1+0)^2}{9+0}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

(d) Note that for $x < 0$ we have $x = -\sqrt{x^2} = -|x|$. For example, $\sqrt{(-5)^2} = \sqrt{25} = 5 = -(-5)$.

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2(1+\frac{4}{x^2})}} = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2}\sqrt{1+\frac{4}{x^2}}} = \lim_{x\to-\infty} \frac{-1}{\sqrt{1+\frac{4}{x^2}}} = \frac{1}{-\sqrt{1+0}} = -1$$

or

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x\to-\infty} \frac{-\sqrt{x^2}}{\sqrt{x^2+4}} = \lim_{x\to-\infty} -\sqrt{\frac{x^2}{x^2+4}} = \lim_{x\to-\infty} -\sqrt{\frac{1}{1+4/x^2}} = -1$$

or

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x\to-\infty} \frac{1}{(\sqrt{x^2+4})/x} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{4}{x^2}}} = \frac{1}{-\sqrt{1+0}} = -1$$

. . . .

(e) *Note that for a limit of type '$\infty - \infty$', it is not correct to conclude that it is 0, since ∞ is not real number. Neither could we say that it will be ∞ nor $-\infty$. Likewise, for a limit of type '$\frac{\infty}{\infty}$', we can't simply cancel the symbol and say that the limit is 1. These types of limits are said to be in 'indeterminate form'. We have to apply some tricks in order to determine the limit.*

$$\lim_{x\to\infty} \sqrt{x^4+6x^2} - x^2 = \lim_{x\to\infty} (\sqrt{x^4+6x^2} - x^2)\left(\frac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2}\right) = \lim_{x\to\infty} \frac{(x^4+6x^2)-x^4}{\sqrt{x^4+6x^2}+x^2}$$

$$= \lim_{x\to\infty} \frac{6x^2}{\sqrt{x^4+6x^2}+x^2} = \lim_{x\to\infty} \frac{6}{\sqrt{1+\frac{6}{x^2}}+1} = \frac{6}{\sqrt{1+0}+1} = 3$$

Note: $\sqrt{x^4+6x^2}+x^2 = \sqrt{x^4(1+\frac{6}{x^2})}+x^2 = \sqrt{x^4}\sqrt{1+\frac{6}{x^2}}+x^2 = x^2\left(\sqrt{1+\frac{6}{x^2}}+1\right)$.

Since $x^2 \geq 0$, we have $\sqrt{x^4} = x^2$.

(f) Dividing both numerator and denominator by e^{3x}, since $\lim\limits_{x\to\infty} e^{3x} = \infty$:

$$\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to\infty} \frac{1-\overbrace{e^{-6x}}^{\to 0}}{1+\underbrace{e^{-6x}}_{\to 0}} = \frac{1-0}{1+0} = 1$$

(g) $\lim\limits_{x\to\infty} [\ln(2+x) - \ln(1+x)]$

<u>Method 1</u> : We use $\ln x$ being a continuous function so that $\lim\limits_{x\to a} \ln(f(x)) = \ln\left(\lim\limits_{x\to a} f(x)\right)$.

$\lim\limits_{x\to\infty} [\ln(2+x) - \ln(1+x)]$

$$= \lim_{x\to\infty} \ln\left(\frac{2+x}{1+x}\right) = \underbrace{\lim_{x\to\infty} \ln\left(\frac{2/x+1}{1/x+1}\right) = \ln\left(\lim_{x\to\infty} \frac{2/x+1}{1/x+1}\right)}_{\ln x \text{ continuous function}} = \ln 1 = 0$$

<u>Method 2</u>:

Using limit law for composite function:

$$\lim_{x\to a} g(x) = b \Rightarrow \lim_{x\to a} f(g(x)) = \lim_{y\to b} f(y).$$

We may replace $g(x)$ by y, and $x \to a$ by $y \to b$.

Note that $\ln(2+x) - \ln(1+x) = \ln\left(\frac{2+x}{1+x}\right)$ which is a composite function of $\ln x$ and $\frac{2+x}{1+x}$.

Let $y = \frac{2+x}{1+x}$ and note that as $x \to \infty$, we have $y = \frac{2+x}{1+x} = \frac{2/x+1}{1/x+1} \to 1$.

Thus, we have $\lim\limits_{x\to\infty} [\ln(2+x) - \ln(1+x)] = \lim\limits_{x\to\infty} \ln\left(\frac{2+x}{1+x}\right) = \lim\limits_{y\to 1} \ln y = \ln 1 = 0$

. . . .

2. Find the domain of $g(x) = \dfrac{\sqrt{x+3}}{x^2-3x-10}$.

[Solution] The value $g(x) = \dfrac{\sqrt{x+3}}{x^2-3x-10}$ is defined if the numerator $\sqrt{x+3}$ is defined and the denominator $x^2-3x-10$ is not zero.

The numerator $\sqrt{x+3}$ is defined for $x+3 \geq 0$, i.e., $x \geq -3$.

The denominator $x^2-3x-10 = (x-5)(x+2)$. It becomes zero whenever $x=-2$ or $x=5$.

Thus, the function g is defined for x satisfying:

$$x \geq -3,\ x \neq -2 \ \&\ x \neq 5.$$

The domain is $[-3,\infty)\backslash\{-2,5\}$.

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3. Use the Intermediate Value Theorem to show that there is a root of the equation $(1-x)^3 = \sin x$ in the interval $(0,1)$.

[Working/Behind the scene:]

We note that: Suppose $c \in (0,1)$ is a root of the equation $(1-x)^3 = \sin x$. Then $(1-c)^3 = \sin c$ which is equivalent to $(1-c)^3 - \sin c = 0$.

This suggests that we use the function $f(x) = (1-x)^3 - \sin x$. We also note that $f(0) = 1 > 0$ and $f(1) = -\sin 1 < 0$. That is, 0 is an intermediate value between $f(0)$ and $f(1)$. So, we shall apply Intermediate Value Theorem to the function $f(x) = (1-x)^3 - \sin x$ on the closed and bounded interval to be $[0,1]$.

(The remaining job is to write the proof:)

[Proof] Let $f(x) = (1-x)^3 - \sin x$ where $x \in [0,1]$. The function f is continuous on $[0,1]$, since it is the difference of two continuous functions $(1-x)^3$ and $\sin x$. Note that

$f(0) > 0$ and $f(1) < 0$.

By the Intermediate Value Theorem, there is a real number $c \in (0,1)$ such that $f(c) = 0$, i.e., c is a root of the equation $(1-x)^3 - \sin x = 0$, i.e., $(1-x)^3 = \sin x$ in the interval $(0,1)$.

CAUTION To apply a theorem or an established result, we have to check/verify that conditions stated in the theorem are satisfied. In the case for Intermediate Value Theorem, we have

(1) to verify (or state it if it is obvious) that the function f is continuous on a closed and bounded interval $[a,b]$;

and

(2) to check that the intermediate value p is between $f(a)$ and $f(b)$.

Once we have checked the conditions are satisfied, we may quote the theorem, and draw the conclusion stated in the theorem. So, we may proceed to say ...

" By the Intermediate Value Theorem, there is a number $c \in (a,b)$ such that $f(c) = p$."

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4. Explain why the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

   [Solution] Let $f(x) = x^3 - 15x + 1$ which is continuous on $[-4, 4]$.

   Computing the values of f at some real numbers as follows:

$$f(-4) = -3 < 0, f(-3) = 19 > 0, f(0) = 1 > 0, f(1) = -13 < 0, f(4) = 5 > 0.$$

   By applying the Intermediate Value Theorem to f on the three intervals $[-4, -3]$, $[0, 1]$ and $[1, 4]$, we conclude that $f(c_1) = 0$ for some $c_1 \in (-4, -3)$, $f(c_2) = 0$ for some $c_2 \in (0, 1)$ and $f(c_3) = 0$ for some $c_3 \in (1, 4)$. This proves that the given equation has three solutions in the interval $[-4, 4]$.

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5. Use the Intermediate Value Theorem to show that the two graphs $y = e^x$ and $y = -x$ intersect.

   [Solution] First we note that suppose the two graphs intersect at $x = c$. That means $e^c = -c$. Rearranging the terms to left hand side: $e^c + c = 0$.

   We set $f(x) = e^x + x$, which is a continuous function on $\mathbb{R}$.

   To find a suitable closed and bounded interval (for applying Intermediate Value Theorem) , we evaluate values of f at some numbers, we look for two values of f with opposite signs so that 0 is an intermediate value.

   Let's do trial and error:

   • $f(0) = 1 > 0$.

   Next, we proceed to find x at which $f(x) < 0$. However, we observe that if we evaluate $f(x)$ at other positive value of x, we will have $f(x) = e^x + x > 0$.

   So, we have to find $x < 0$ at which $f(x) < 0$.

   • $f(-1) = e^{-1} - 1 < 0$

   (Without using a calculating machine (but with a mathematical mind!) , note that $e^{-1} = 1/e < 1$ so that $e^{-1} - 1 < 0$.)

   This suggests we may choose the interval $[-1, 0]$.

   Now, consider $f(x) = e^x + x$ on $[-1, 0]$. Note that f is continuous on $[-1, 0]$, since it is a sum of continuous functions e^x and x. Furthermore, we have $f(-1) = e^{-1} - 1 < 0 < f(0) = 1$. By Intermediate Value Theorem, we conclude that there is a real number $c \in (-1, 0)$ such that $f(c) = 0$.

   Therefore, the graphs $y = e^x$ and $y = -x$ intersect at $x = c$.

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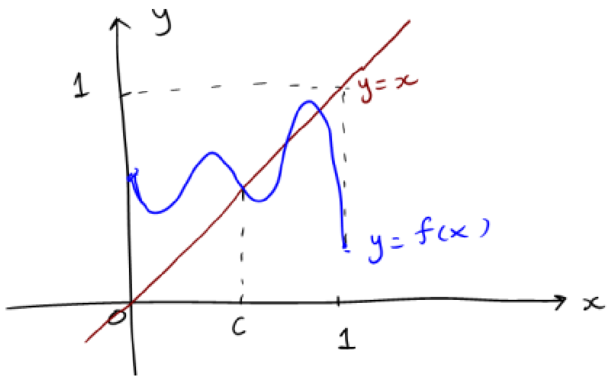
6. Suppose that a function f is continuous on the closed interval $[0, 1]$ and $0 \leq f(x) \leq 1$ for every $x \in [0, 1]$. Is it true that $f(c) = c$ for some $c \in [0, 1]$? Justify your answer.

   Remark Note that $f(c) = c$ means the graphs of $y = f(x)$ and $y = x$ intersect at $x = c$. The condition $0 \leq f(x) \leq 1$ means that graph of $y = f(x)$ lies within the 'box' $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

   To determine whether the statement is true or false, we may sketch a continuous curve representing the graph of a continuous function $y = f(x)$ within the 'box' $0 \leq x \leq 1$ and $0 \leq y \leq 1$.



   We observe that the curve will intersect the line $y = x$ at some $c \in [0, 1]$.

   This provides a clue that the result is true. We shall now proceed to justify it by giving a proof.

   [Solution] Note that $f(c) = c$ is equivalent to $f(c) - c = 0$.

   Let $g(x) = f(x) - x$ where $x \in [0, 1]$.

   We shall prove that $g(c) = 0$ for some $c \in [0, 1]$.

   Note that $g(x)$ is continuous as it is a sum of continuous functions $f(x)$ and x on $[0, 1]$.

   We evaluate values of g at end-points:

   Given that $0 \leq f(x) \leq 1$, we have, in particular, with $x = 0$ and $x = 1$, $0 \leq f(0) \leq 1$ and $0 \leq f(1) \leq 1$.

   It follows that $g(0) = f(0) - 0 = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$.

   We consider the following three possible cases:

   (i) Case $g(0) = 0$:
   In this case, we have $f(0) = 0$, i.e., $c = 0$.

   (ii) Case $g(1) = 0$:
   In this case, we have $f(1) = 1$, i.e., $c = 1$.

   (iii) Case $g(0) > 0$ and $g(1) < 0$:
   For this case, we apply Intermediate Value Theorem to $g(x)$ on $[0, 1]$, since $g(x)$ is continuous on $[0, 1$, and $g(1) < 0 < g(0)$.
   We conclude that there is a real number $c \in (0, 1)$ such that $g(c) = 0$, i.e., $f(c) = c$.

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7. (a) Write down the definition of $f'(0)$ and use it to find $f'(0)$ where $f(x) = 1 - x^3$.
   (b) Use the value $f'(0)$ you have found in part (a) to write down an equation of the tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$.

[Solution]

(a) $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{f(h) - f(0)}{h}$ (if this limit exits) .

For $f(x) = 1 - x^3$, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{(1 - h^3) - 1}{h} = \lim_{h\to 0} -h^2 = 0.$$

(b) An equation of the tangent line to the curve $y = 1-x^3$ at the point $(0, 1)$ is given by $\dfrac{y - f(0)}{x - 0} = f'(0)$, which gives $y = 1$.

Remark In general, the equation of a tangent to the graph $y = f(x)$ at a point with coordinate $(a, f(a))$ is given by the equation $y - f(a) = f'(a)(x - a)$.

. . . .

8. Use the <u>definition of derivative</u> to find the first derivative of the following functions:

   (a) $f(x) = \dfrac{1}{5 - 3x}$
   (b) $g(x) = \sqrt{x^2 + 3}$.

[Solution]

(a) $f(x) = \dfrac{1}{5 - 3x}$

By the definition of derivative, we compute the following limit

$$f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x} = \lim_{t\to x} \frac{\frac{1}{5-3t} - \frac{1}{5-3x}}{t - x} = \lim_{t\to x} \frac{(5 - 3x) - (5 - 3t)}{(5 - 3t)(5 - 3x)(t - x)}$$

$$= \lim_{t\to x} \frac{3(t - x)}{(5 - 3t)(5 - 3x)(t - x)} = \lim_{t\to x} \frac{3}{(5 - 3t)(5 - 3x)} = \frac{3}{(5 - 3x)^2}$$

Alternatively, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{5-3(x+h)} - \frac{1}{5-3x}}{h} = \lim_{h\to 0} \frac{(5 - 3x) - (5 - 3(x + h))}{(5 - 3(x + h))(5 - 3x)(h)}$$

$$= \lim_{h\to 0} \frac{3h}{(5 - 3(x + h))(5 - 3x)h} = \lim_{h\to 0} \frac{3}{(5 - 3(x + h))(5 - 3x)} = \frac{3}{(5 - 3x)^2}$$

(b) $g(x) = \sqrt{x^2 + 3}$.

By the definition of derivative, we compute the following limit

$$g'(x) = \lim_{t\to x} \frac{g(t) - g(x)}{t - x} = \lim_{t\to x} \frac{\sqrt{t^2 + 3} - \sqrt{x^2 + 3}}{t - x} = \lim_{t\to x} \frac{(t^2 + 3) - (x^2 + 3)}{(t - x)(\sqrt{t^2 + 3} + \sqrt{x^2 + 3})}$$

$$= \lim_{t\to x} \frac{t^2 - x^2}{(t - x)(\sqrt{t^2 + 3} + \sqrt{x^2 + 3})} = \lim_{t\to x} \frac{(t + x)}{(\sqrt{t^2 + 3} + \sqrt{x^2 + 3})} = \frac{2x}{2\sqrt{x^2 + 3}} = \frac{x}{\sqrt{x^2 + 3}}$$

Alternatively, we have

$$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+3}-\sqrt{x^2+3}}{h} = \lim_{h\to 0} \frac{((x+h)^2+3)-(x^2+3)}{h(\sqrt{(x+h)^2+3}+\sqrt{x^2+3})}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2)-x^2}{h(\sqrt{(x+h)^2+3}+\sqrt{x^2+3})} = \lim_{h\to 0} \frac{2x+h}{(\sqrt{(x+h)^2+3}+\sqrt{x^2+3})} = \frac{2x}{2\sqrt{x^2+3}} = \frac{x}{\sqrt{x^2+3}}$$

....

9. Consider the function f where $f(x) = x|x|$. Is f differentiable at $x = 0$? If it is, determine $f'(0)$.

We provide three different solutions.

[Solution 1] To check whether f is differentiable we have to check whether the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists.

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x|x|-0}{x} = \lim_{x\to 0} \frac{x|x|}{x} = \lim_{x\to 0} |x| = 0$$

Therefore, f is differentiable at $x = 0$ and $f'(x) = 0$.

[Solution 2] Since $|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0; \end{cases}$, we re-write $f(x)$ as follows:

$$f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0; \\ -x^2 & \text{if } x < 0; \end{cases}$$

Note that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{f(x)-0}{x} = \lim_{x\to 0} \frac{f(x)}{x}$ which is determined via one-sided limits.

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x^2}{x} = \lim_{x\to 0^+} x = 0$$

$$\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{x(-x)}{x} = \lim_{x\to 0^+} -x = 0$$

By Equal One-sided Theorem, $\lim_{x\to 0} \frac{f(x)}{x} = 0$, which means that $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$.
Therefore f is differentiable at 0 and $f'(0) = 0$.

[Solution 3 Using the Theorem in the Lecture] The function $f(x) = x|x|$ is continuous at $x = 0$ since it is a product of x and $|x|$ where both x and $|x|$ are continuous at $x = 0$.
From

$$f(x) = x|x| = \begin{cases} = x(x) = x^2 & \text{if } x \geq 0; \\ = x(-x) = -x^2 & \text{if } x < 0, \end{cases}$$

we have for $x \neq 0$,

$$f'(x) = \begin{cases} 2x & \text{if } x > 0; \\ -2x & \text{if } x < 0. \end{cases}$$

Since $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} 2x = 0$ and $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} -2x = 0$, we conclude that f is differentiable at $x = 0$ and $f'(0) = 0$.

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10. Consider f which is defined as follows:

$$f(x) = \begin{cases} \dfrac{e^x}{2+x} & \text{if } x \geq 0, \\ \cos(1 - e^{\pi x}) & \text{if } x < 0. \end{cases}$$

Is f differentiable at $x = 0$?

[Solution]

Recall: f is differentiable at $x = c$ implies f is continuous at $x = c$.

Thus, we have

f is NOT continuous at $x = c$ implies f is NOT differentiable at $x = c$.

We check whether f is continuous at $x = 0$,i.e., whether $\lim_{x \to 0} f(x) = f(0)$?

Note that $f(0) = \dfrac{e^0}{2+0} = \dfrac{1}{2}$. To find $\lim_{x \to 0} f(x)$, we check one-sided limits at $x = 0$ as follows.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^x}{2+x} = \frac{1}{2}$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \cos(1 - e^{\pi x}) = \cos(1 - e^0) = \cos 0 = 1$$

Since $\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$, the limit $\lim_{x \to 0} f(x)$ does not exist.

Therefore, the function f is not continuous at $x = 0$. Hence it is not differentiable at $x = 0$.

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