Nanyang Technological University
SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 7**

1. Suppose f is differentiable and $f(x) > 0$.

   Use the following definition of derivative, $g'(x) = \lim_{t \to x} \frac{g(t) - g(x)}{t - x}$, to prove that

   (a) $\frac{d}{dx}(179 f(x)) = 179 f'(x)$.
   [Proof]

   $$\frac{d}{dx}(179 f(x)) = \lim_{t \to x} \frac{179 f(t) - 179 f(x)}{t - x} = 179 \left(\lim_{t \to x} \frac{f(t) - f(x)}{t - x} \right) = 179 f'(x)$$

   (b) $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$.
   [Proof]
   $\frac{d}{dx}\sqrt{f(x)} = \lim_{t \to x} \frac{\sqrt{f(t)} - \sqrt{f(x)}}{t - x} = \lim_{t \to x} \frac{(\sqrt{f(t)} - \sqrt{f(x)})(\sqrt{f(t)} - \sqrt{f(x)})}{(t - x)(\sqrt{f(t)} + \sqrt{f(x)})}$
   $= \lim_{t \to x} \frac{f(t) - f(x)}{(t - x)\left(\sqrt{f(t)} + \sqrt{f(x)}\right)}$
   $= \lim_{t \to x} \frac{f(t) - f(x)}{(t - x)} \lim_{t \to x} \frac{1}{\sqrt{f(t)} + \sqrt{f(x)}}$
   $= f'(x) \cdot \frac{1}{\sqrt{f(x)} + \sqrt{f(x)}} = \frac{f'(x)}{2\sqrt{x}}$
   Note that as f is differentiable at x, it is continuous at x too so that

   $$\lim_{t \to x} \frac{1}{\sqrt{f(t)} + \sqrt{f(x)}} = \frac{1}{2\sqrt{x}}.$$

   (c) $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{(f(x))^2}$.
   [Proof.]

   $$\begin{aligned} \lim_{t \to x} \frac{\frac{1}{f(t)} - \frac{1}{f(x)}}{t - x} &= \lim_{t \to x} \frac{f(x) - f(t)}{f(t) \cdot f(x) \cdot (t - x)} \\ &= \lim_{t \to x} \frac{-(f(t) - f(x))}{t - x} \cdot \frac{1}{f(t) \cdot f(x)} \\ &= \frac{-f'(x)}{(f(x))^2} \end{aligned}$$

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2. If $r(t) = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is $\frac{dr}{dt}$ at $t = 0$?

   [Solution] By Chain Rule, we have $\frac{dr}{dt} = f'(t) \cdot \cos(f(t))$.

   At $t = 0$, $\frac{dr}{dt} = f'(0) \cdot \cos(f(0)) = 4\cos(\pi/3) = 2$.

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3. Calculate y'.

(a) $y = \cos(\tan x)$

(b) $y = \left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}}$

(c) $y = \dfrac{1}{\sin(x - \sin x)}$

(d) $x^2 \cos y + \sin 2y = xy$

(e) $x \tan y = y - 1$

(f) $y = \ln(\sec x)$

(g) $y = \ln(\sec x + \tan x)$

(h) $y = \sin^{-1}(1 - x)$

[Solution]

(a) $y = \cos(\tan x)$: $y' = -(\sec^2 x)\sin(\tan x)$.

(b) $y = \left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}}$: $y' = \sqrt{7}(1 - \dfrac{2}{x^3})\left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}-1}$

(c) $y = \dfrac{1}{\sin(x - \sin x)}$: $y' = \dfrac{-(\cos(x - \sin x))(1 - \cos x)}{\sin^2(x - \sin x)}$

(d) $x^2 \cos y + \sin 2y = xy$

[Solution] Differentiating w.r.t x:

$$2x\cos y \ + x^2(-\sin y)\frac{dy}{dx} + 2\cos(2y)\frac{dy}{dx} = y + x\frac{dy}{dx}.$$

Therefore,

$$\left(x^2 \sin y - 2\cos(2y) + x\right)\frac{dy}{dx} = 2x\cos y - y$$

and hence

$$y' = \frac{2x\cos y - y}{x^2 \sin y - 2\cos(2y) + x}$$

(e) $x \tan y = y - 1$

[Solution] Differentiating w.r.t x: $\tan y + x\sec^2 y\dfrac{dy}{dx} = \dfrac{dy}{dx}$. Thus,

$$\frac{dy}{dx} = \frac{\tan y}{1 - x\sec^2 y}.$$

(f) $y = \ln(\sec x)$: $y' = \dfrac{\sec x \tan x}{\sec x} = \tan x$

(g) $y = \ln(\sec x + \tan x)$: $y' = \dfrac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$

(h) $y = \sin^{-1}(1 - x)$: $\dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1 - (1 - x)^2}}$

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4. Find the second derivative $f''(x)$ of $f(x) = \dfrac{x}{1+x^2}$.

[Solution]

$f'(x) = \dfrac{(1+x^2) - 2x^2}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$

$f''(x) = \dfrac{-2x(1+x^2)^2 - (1-x^2)(4x)(1+x^2)}{(1+x^2)^4} = \dfrac{-2x(3-x^2)}{(1+x^2)^3}$

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5. Find $f'(x)$.

(a) $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right)$: $\quad f'(x) = \dfrac{-1}{(\ln\ 10)x(x-1)}$

(b) $f(x) = \left(\dfrac{1+\ln x}{1-\ln x}\right) = -1 + \dfrac{2}{1-\ln x}$

Thus, we have $f'(x) = \dfrac{2}{x(1-\ln x)^2}$

(c) $f(x) = x\ln(1+e^x)$: $\ f'(x) = \ln(1+e^x) + \dfrac{xe^x}{1+e^x}$

(d) $f(x) = (\ln(1+e^x))^2$: $\ f'(x) = \dfrac{2e^x\ln(1+e^x)}{1+e^x}$

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6. Find an equation of the tangent line to the curve $y = \dfrac{e^x}{x}$ at the point (i) $(1, e)$, (ii) where $x = -1$.

[Solution] Note that $\dfrac{dy}{dx} = \dfrac{e^x(x-1)}{x^2}$

(i) At $(1, e)$, we have $\dfrac{dy}{dx}\Big|_{x=1} = 0$. (Horizontal Tangent)

Thus the equation of the tangent line is $y = e$.

Alternatively, using the equation for tangent, we have

$$y - e = (x-1)\left(\frac{dy}{dx}\Big|_{x=1}\right), \text{ i.e., } y = e.$$

(ii) AT $x = -1$, we have $y = -1/e$ and $\dfrac{dy}{dx}\Big|_{x=-1} = -2/e$.

The equation of the tangent at $x = -1$ is

$$y - (-1/e) = (x - (-1))(-2/e), \text{ i.e., } y = (-2/e)x - 3/e.$$

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7. If n is a positive number, prove that

$$\frac{d}{dx}(\sin^n x \cos nx) = n\sin^{n-1} x \cos(n+1)x$$

[Solution] Use Product Rule:

$$\frac{d}{dx}(\sin^n x\cos nx) = n\left(\sin^{n-1} x\right)(\cos x)\cos nx + \sin^n x(-n\sin nx)$$

$$= n\left(\sin^{n-1} x\right)(\cos x\cos nx - \sin x\sin nx) = n\sin^{n-1} x\ \cos(n+1)x$$

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8. (a) Use implicit differentiation to prove that

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$$

(b) Use the formula established in part (a) to find $\dfrac{dy}{dx}$ for
(i) $y = x\tan^{-1}\left(\frac{x}{2}\right)$, (ii) $y = \tan^{-1}(\ln x)$ and (iii) $\tan^{-1}(xy) = 1 + x^2y$.

[Solution]

(a) Let $y = \tan^{-1} x$.Then $\tan y = x$.
Differentiating with respect to x yields

$$\sec^2 y\frac{dy}{dx} = 1, \text{ which gives}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

(b) (i) $\dfrac{dy}{dx} = \tan^{-1}\left(\dfrac{x}{2}\right) + x\left(\dfrac{1}{1+(x/2)^2}\right)(\dfrac{1}{2}) = \tan^{-1}\left(\dfrac{x}{2}\right) + \dfrac{2x}{4+x^2}$

(ii) $\dfrac{dy}{dx} = \left(\dfrac{1}{1+(\ln x)^2}\right)(\dfrac{1}{x}) = \dfrac{1}{x(1+(\ln x)^2)}$

(iii) Differentiating the given equation implicitly w.r.t. x:

$$\frac{1}{1+(xy)^2}\left(y + x\frac{dy}{dx}\right) = 2xy + x^2\frac{dy}{dx}$$

which gives

$$\frac{y}{1+(xy)^2} + \frac{x}{1+(xy)^2}\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx}$$

$$\left(\frac{x}{1+(xy)^2} - x^2\right)\frac{dy}{dx} = 2xy - \frac{y}{1+(xy)^2}$$

$$\left(x - x^2(1+(xy)^2)\right)\frac{dy}{dx} = 2xy + 2(xy)^3 - y$$

$$\frac{dy}{dx} = \frac{2xy + 2(xy)^3 - y}{(x - x^2(1+(xy)^2))}$$

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9. Find the derivative of the following function

$$f(x) = (\ln x)^{\cos x}, x > 1.$$

[Solution] Applying logarithmic function to both sides of the given equations:

$$\ln y = \ln\left((\ln x)^{\cos x}\right) = (\cos x)\ln(\ln x)$$

Differentiating with respect to x implicitly,

$$\frac{1}{y}y' = (-\sin x)\ln(\ln x) + (\cos x)\cdot\frac{1}{\ln x}\cdot(\frac{1}{x}) = (-\sin x)\ln(\ln x) + \frac{\cos x}{x\ln x}$$

Thus, we have

$$y' = y\left((-\sin x)\ln(\ln x) + \frac{\cos x}{x\ln x}\right) = (\ln x)^{\cos x}\left((-\sin x)\ln(\ln x) + \frac{\cos x}{x\ln x}\right)$$

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10. (Thomas' Calculus, Exercise 3.8, Q 16) The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

    (a) How are $\frac{dP}{dt}$, $\frac{dR}{dt}$ and $\frac{dI}{dt}$ related if P, R and I are functions of t?

    (b) How is $\frac{dR}{dt}$ related to $\frac{dI}{dt}$ if $P = P_0$ is constant?

    [Solution]

    (a)
$$\frac{dP}{dt} = I^2\frac{dR}{dt} + 2RI\frac{dI}{dt}.$$

    (b) If $P = P_0$ is constant, then we have $\frac{dP}{dt} = 0$. Using part(a) and $P_0 = RI^2$, we have
$$\frac{dR}{dt} = -\frac{2R}{I}\frac{dI}{dt} = -\frac{2P_0}{I^3}\frac{dI}{dt}.$$

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11. (Thomas' Calculus, Exercise 7.7, Q 78 a) (**Accelerations whose magnitudes are propositional to displacement**) Suppose that the position of a body moving along a coordinate line at time t is $s = a\cos kt + b\sin kt$. Show that the acceleration $\frac{d^2s}{dt^2}$ is proportional to s and it is directed to the origin.

    [Solution]
$$s = a\cos kt + b\sin kt, \frac{ds}{dt} = -ka\sin kt + kb\cos kt,$$
$$\frac{d^2s}{dt^2} = -k^2a\cos kt - k^2b\sin kt = -k^2\left(a\cos kt + b\sin kt\right) = -k^2s,$$
    which shows that the acceleration $\frac{d^2s}{dt^2}$ is proportional to s and it is directed to the origin (from the negative sign).

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12. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

    [Solution] Let x cm be the diameter of the snowball, and $S\text{cm}^2/$ be its surface area.
    The information given tells us that when $x = 10$, we have $\frac{dS}{dt} = -1$, we are required to find $\frac{dx}{dt}$.
    Note that the surface area $S\text{cm}^2/$ of the snow ball is given by $S = 4\pi(x/2)^2 = \pi x^2$.
    Differentiating with respect to t:
$$\frac{dS}{dt} = \pi 2x\frac{dx}{dt}. \quad (*)$$
    AT $x = 10$, we have $\frac{dS}{dt} = -1$, so that (*) gives
$$-1 = \pi 2(10)\frac{dx}{dt}.$$
    which gives
$$\frac{dx}{dt} = -\frac{1}{20\pi} \approx -0.0159.$$

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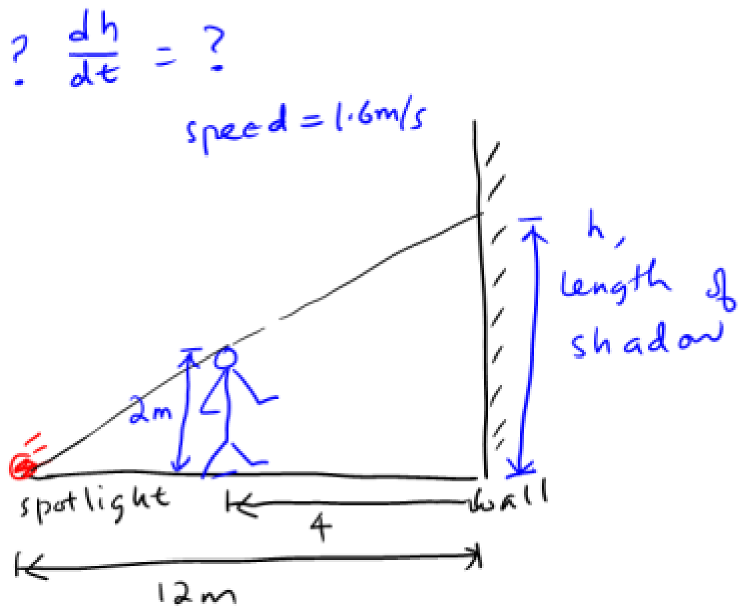
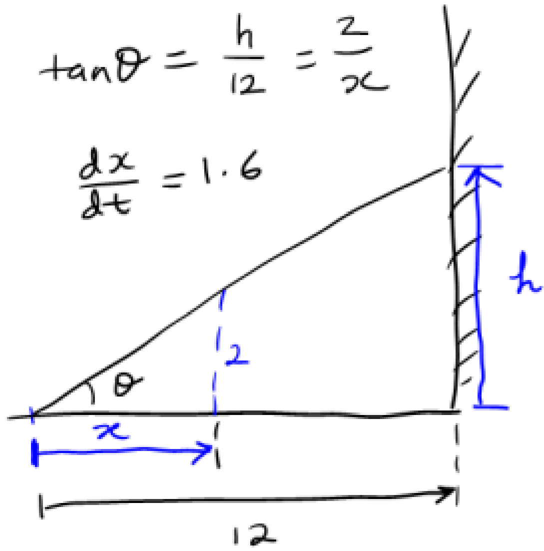
13. A spotlight on the ground shines on a wall 12 m away. If a 2 m tall man walks from the spotlight straight towards the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing, at the moment when he is 4 m from the building?





[Solution] Letting h m denote the height of the shadow on the wall, and x m the distance the man has walked from the spot light to the wall, using an argument with similar triangles, we have

$$\frac{h}{12} = \frac{2}{x}.$$

Both h and x are dependent on time t. Differentiating the above equation w.r.t. t we have

$$\frac{\frac{dh}{dt}}{12} = \frac{-2\frac{dx}{dt}}{x^2}.$$

Substituting $\frac{dx}{dt} = 1.6$ m/s and $x = 8$ m (since the man is 4m from the building), we get

$$\frac{dh}{dt} = \frac{-24\frac{dx}{dt}}{x^2} = \frac{-24 \cdot (1.6)}{8^2} \text{ m/s} = -\frac{3}{5} \text{ m/s}.$$

Hence, at this moment, the shadow is getting shorter at a rate of 0.6 m/s.

(Note: The negative sign in $\frac{dh}{dt}$ indicates that the height of the shadow is decreasing.)

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14. (Thomas' Calculus, Exercise 3.8, Q36) A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$m?

[Solution] Let $P(x, y)$ represent a point on the curve $y = x^2$. Then

$$\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$$

Differentiating with respect to t yields

$$\sec^2\theta \frac{d\theta}{dt} = \frac{dx}{dt}.$$

Hence we have,

$$\frac{d\theta}{dt} = \cos^2\theta \frac{dx}{dt}$$

Since $\frac{dx}{dt} = 10$ m/sec and $\cos^2\theta = \frac{x^2}{x^2+y^2} = \frac{3^2}{3^2+9^2} = \frac{1}{10}$ when $x = 3$, we have

$$\frac{d\theta}{dt} = \cos^2\theta \frac{dx}{dt} = 1 \text{ rad/sec.}$$

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