Nanyang Technological University

SPMS/Division of Mathematical Sciences

2015/16 Semester 1 **MH1810 Mathematics I** **Tutorial 8**

Topics: Linearization, Differentials, Newton's Method, Closed Interval Method.

1. Use the result $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to determine the following limits.

   (a) $\lim_{x\to 0} \frac{\sin x^2}{x^2}$

   (b) $\lim_{x\to 0} \frac{\sin 3x}{x}$

   (c) $\lim_{x\to 0} \frac{\sin x^2}{\sin x}$

   (d) $\lim_{x\to \pi} \frac{\sin(x-\pi)}{x-\pi}$

   (e) $\lim_{x\to 0} \frac{\sin(\sin x)}{x}$

   (f) $\lim_{x\to \infty} x\sin(\frac{1}{x})$

   [SOLUTION]

   (a) $\lim_{x\to 0} \frac{\sin x^2}{x^2} = \lim_{y\to 0} \frac{\sin y}{y} = 1$. (replace $y = x^2$)

   (b) $\lim_{x\to 0} \frac{\sin 3x}{x} = \underbrace{\lim_{x\to 0} 3\frac{\sin 3x}{3x}}_{*} = 3\lim_{x\to 0} \frac{\sin 3x}{3x} = 3\underbrace{\lim_{y\to 0} \frac{\sin y}{y}}_{y=3x} = 3(1) = 3.$

   NOTE: (*) We want to use $g(x) = 3x$ since we have $\sin 3x$ in the numerator. So, we need the denominator to be $3x$. Balance it by multiplying a 3 to the numerator.

   (c) $\lim_{x\to 0} \frac{\sin x^2}{\sin x} = \lim_{x\to 0} \frac{\sin x^2}{x^2} \cdot \frac{x}{\sin x} \cdot \frac{x^2}{x} = \lim_{x\to 0} \frac{\sin x^2}{x^2} \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} \lim_{x\to 0} \frac{x^2}{x} = 1 \cdot 1 \cdot 0 = 0$

   (d) Note that $\lim_{x\to\pi}(x-\pi) = 0$ and $x - \pi \neq 0$ if $x \neq \pi$.

   $$\lim_{x\to \pi} \frac{\sin(x-\pi)}{x-\pi} = \lim_{y\to 0} \frac{\sin y}{y} = 1$$

   (e) $\lim_{x\to 0} \frac{\sin(\sin x)}{x} = \lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}\frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin(\sin x)}{\sin x} \lim_{x\to 0} \frac{\sin x}{x} = 1(1) = 1$

   (f) $\lim_{x\to \infty} x\sin(\frac{1}{x}) = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{1/x} = \lim_{y\to 0^+} \frac{\sin y}{y} = 1$

   (Let $y = \frac{1}{x}$. Note that $x \to \infty$ implies that $y \to 0^+$.)

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2. For the function $f(x)$, write down the linearization $L(x) = f(a) + (x-a)f'(a)$ at $x = a$.

   (a) $f(x) = \sqrt{x^2+9}$, $a = -4$

(b) $f(x) = \tan x$, $a = \pi$

[Solution]

(a) $f(x) = \sqrt{x^2+9}$ and $f'(x) = \dfrac{x}{\sqrt{x^2+9}}$.
At $a = -4$, $f(-4) = 5$ and $f'(-4) = -4/5$ so that

$$L(x) = 5 - \frac{4}{5}(x+4).$$

(b) $f(x) = \tan x$ and $f'(x) = \sec^2 x$.
Thus, at $x = \pi$, we have $f(\pi) = 0$ and $f'(\pi) = 1$.
Hence, we have

$$L(x) = (x - \pi).$$

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3. Use the linearization of a suitable function $f(x)$ at a point $x = a$ to approximate $\sqrt[3]{7.99}$.
[Solution] We shall use the linearization of $f(x) = \sqrt[3]{x}$ at $x = 8$ to approximate $\sqrt[3]{7.99}$.
We have $f(8) = 2$ and $f'(x) = \dfrac{1}{3}x^{\frac{-2}{3}}$, so that $f'(8) = \dfrac{1}{3}8^{\frac{-2}{3}} = \dfrac{1}{12}$.
Thus,

$$L(x) = f(8) + (x-8)f'(8) = 2 + \frac{(x-8)}{12}.$$

Hence, we have

$$\sqrt[3]{7.99} \approx L(7.99) = 2 + \frac{(7.99-8)}{12} = 2 - \frac{1}{1200}.$$

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4. The function $f(x) = 2x^2 + 4x - 3$ changes value when x changes from $x_0 = -1$ to $x_1 = -0.9$. Find

(a) the change $\triangle f = f(x_1) - f(x_0)$;
(b) the value of the estimate $df = f'(x_0)dx$, where $dx = x_1 - x_0$; and
(c) the approximation error $|\triangle f - df|$.

[Answers] Note that $f(x) = 2x^2 + 4x - 3$ and $f'(x) = 4x + 4$.
(a) 0.02 (b) 0 (c) 0.02

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5. Write a differential formula that estimates the given change in the Volume $V = \pi r^2 h$ of a right circular cylinder when the height changes from h_0 to $h_0 + dh$ and the radius does not change.
[Solution]

$$S'(h) = \pi r^2$$

Thus, we have

$$dV = (V'(h_0)) \cdot (dh) = \pi r^2 (dh).$$

For example, when $r = 4$ cm and $h_0 = 0.1$ cm and $dh = 0.01$ cm, we have

$$dV = \pi r^2 (dh) = 0.16\pi \text{ (cm}^2\text{)}.$$

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6. Use Newton's method to estimate $\sqrt[4]{2}$, the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.

   [Solution] Let $f(x) = x^4 - 2$. Then $f'(x) = 4x^3$.

   The Newton's iterates are
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 2}{4x_n^3}.$$

   When $n = 0$, we have $x_0 = 1$, and we compute
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{4} = 1.25;$$

   and
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.25 - \frac{(1.25)^4 - 2}{4(1.25^3)} = 1.1935.$$

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7. Use the Intermediate Value Theorem to show that $f(x) = x^3 + 2x - 4$ has a root in $(1, 2)$. Then use Newton's Method to find the root to five decimal places. (Answer: 1.17951)

   [Solution] The function $f(x) = x^3 + 2x - 4$ is continuous on $[1, 2]$. Note that $f(1) = -1 < 0$ and $f(2) = 8 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for some $c \in (1, 2)$.

   The Newton's iterates are
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   We have $f(x) = x^3 + 2x - 4$ and $f'(x) = 3x^2 + 2$.

   When $n = 0$, take $x_0 = 1$. Then
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{5} = 1.2$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.2 - \frac{f(1.2)}{f'(1.2)} = 1.2 - \frac{0.128}{6.32} = 1.179747$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.179509$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 1.179509$$

   Approximated Solution is $x = 1.179509$.

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8. Use the closed interval method to find the global (absolute) maximum and minimum values of $f(x) = \sqrt{4 - x^2}$ on the interval $[-2, 1]$.

   (Answer: Global maximum: 2, global minimum: 0)

   [Solution] The function $f(x) = \sqrt{4 - x^2}$ is continuous on the interval $[-2, 1]$. Thus, it has a global maximum and minimum values.

   Its derivative is $f'(x) = \dfrac{-x}{\sqrt{4 - x^2}}$.

Thus, f has not singular point in $(-2, 1)$. It remains to find all stationary points points of f on $(-2, 1)$. Setting $f'(x) = 0$, we have $x = 0$ as the only stationary point.

Comparing values of f at critical points and end-points:

$$f(-2) = 0, \quad f(0) = 2, \quad f(1) = \sqrt{3}.$$

Thus, the global maximum of f is $f(0) = 2$ and its global minimum is $f(-2) = 0$.

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9. Find the global maximum and minimum values of $f(x) = e^x - 2x$ on $[0, 1]$.

   [Solution] Since $f(x)$ is continuous on $[0, 1]$, we use closed-interval method.

   (1) $f'(x) = e^x - 2$ at every $x \in (0, 1)$.

   – No singular points.

   – Stationary points: $f'(x) = 0 \iff x = \ln 2$.

   (2) End points $x = 0$ and $x = 1$

   (3) Comparing $f(0) = 1$, $f(\ln 2) = 2 - 2\ln 2 =$ and $f(1) = e - 2$, we have

   $$f(\ln 2) < f(1) < f(0).$$

   Therefore the global maximum is $f(0) = 1$ and

   the global minimum is $f(\ln 2) = 2 - 2\ln 2$.

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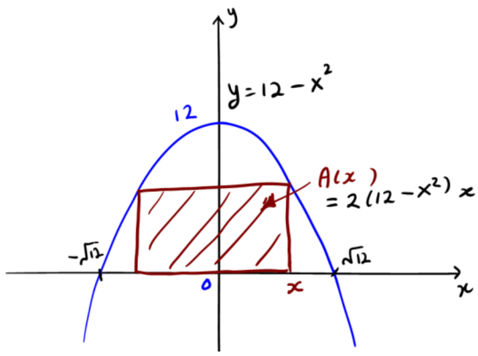
10. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?

    [Solution]



Note that the area $A(x)$ of the rectangle is $A(x) = 2x(12 - x^2)$ where $x \in [0, \sqrt{12}]$.

We use Closed-interval method.

To determine all critical points in $(0, \sqrt{12})$, we first note that $A'(x) = 24 - 6x^2$. Thus, there is no singular point.

For stationary points, we set $A'(x) = 0$, i.e., $24 - 6x^2 = 0$, which give $x = 2$ or -2 (rejected).

Comparing values of $A(x)$ at end-points and critical points:

$$A(0) = 0, \ A(2) = 32 \text{ and } A\left(\sqrt{12}\right) = 0,$$

we conclude that $A(2) = 32$ is the global maximum of A and the rectangle has dimension 4 units by 8 units.

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