NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2012–2013

MH1810 – Mathematics I

NOVEMBER 2012 TIME ALLOWED: 2.5 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **EIGHT (8)** questions and comprises **EIGHT (8)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the attachments.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1. **(10 Marks)**

Let $\alpha = 1 + \sqrt{3}i$ and $\beta = \frac{1}{2} - \frac{i}{2}$.

(i) Express the complex numbers α and β in trigonometric form, $r\,(\cos\theta + i\sin\theta)$, where $r > 0$ and $\theta \in (-\pi, \pi]$

(ii) Simplify $\alpha^3\beta^4$, in the form $x + iy$.

(iii) Find a non-zero polynomial $P\,(z)$ with real coefficients such that

$$P\,(\alpha) = P\,(\beta) = 0.$$

QUESTION 2. **(10 Marks)**

A movie theatre screen which is 18 m tall, has its bottom edge 2 m above an observer's eye. The visual angle θ of the viewer x m away from the foot of the wall is the difference between the angle of elevation to the top edge and the bottom edge (see diagram).

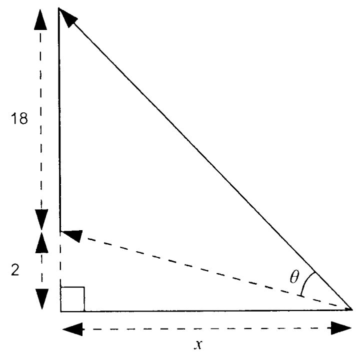
(i) Show that

$$\theta = \tan^{-1}\frac{20}{x} - \tan^{-1}\frac{2}{x}, \text{ for } x > 0$$

and

$$\frac{d\theta}{dx} = \frac{-18(x^2 - 40)}{(x^2 + 400)(x^2 + 4)}.$$

(ii) Determine the distance the viewer must sit to obtain the maximal visual angle. Justify your answer.



QUESTION 3. **(15 Marks)**

(a) Consider a plane $\prod$ through points $A(1,1,0)$, $B(-1,0,-1)$ and $C(0,-1,2)$ in $\mathbb{R}^3$.

(i) Find two real numbers s and t such that the vector $\mathbf{n} = \begin{pmatrix} 1 \\ s \\ t \end{pmatrix}$ is a normal vector to the plane $\prod$.

(ii) Find the distance of the plane $\prod$ from the origin $O(0,0,0)$.

(b) Consider the following system of linear equations

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & c & = & 1 \\ -a & + & 4b & + & 2c & = & 0 \\ a & + & rb & - & c & = & -1 \end{array}$$

(i) Find the values of r at which Cramer's rule is applicable.

(ii) For $r = 1$, use Cramer's Rule to determine the unknown b.

QUESTION 4. **(12 Marks)**

Consider the function f defined as follows:

$$f(x) = \begin{cases} \dfrac{x^3 + \sinh x}{e^{3x} - \cos x} & \text{if } x < 0, \\ 3 & \text{if } x = 0, \\ \sqrt{x^6 + \sin^2 x} - x^3 & \text{if } x > 0. \end{cases}$$

Determine whether each of the following limits exists. Justify your answer. If it exists, what is its value?

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to \infty} f(x)$

QUESTION 5. **(15 Marks)**

Recall that

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2} \text{ and } \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

(a) Prove that $\cosh^2 x - \sinh^2 x = 1$.

(b) Prove that

$$\frac{d}{dx}(\sinh x) = \cosh x.$$

(c) Prove that

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}.$$

(d) Evaluate the indefinite integral

$$\int x^2 \sinh^{-1} x \, dx.$$

QUESTION 6 **(12 Marks)**

(a) Prove that if f is differentiable at $x = c$, then f is continuous at $x = c$.

(b) Find the exact volume of the solid when the region bounded by the curve $y = \sin(x^2)$ and lines $x = 0, x = \sqrt{\pi}$ and $y = 0$ is rotated about the y-axis.

Question 7 **(12 Marks)**

(a) By considering the derivative, explain why the polynomial

$$P(x) = x^3 - 3x^2 + 18x + 2$$

has at most one real root.

(b) Show that P has a root in $[-1, 0]$.

(c) Use Newton's method, with $x_0 = 0$, to approximate the root of P in $[-1, 0]$ by the second iterate, x_2.

QUESTION 8 **(14 Marks)**

(a) Evaluate the improper integral

$$\int_0^\infty \frac{1}{x^2 - 6x + 13}\, dx.$$

(b) Evaluate

$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{n+n}} \right]$$

by expressing it as a definite integral $\int_0^1 f(x)\, dx$ for some function f.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x - a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\mathrm{csch}^2 x$$

$$\frac{d}{dx}(\mathrm{sech}\ x) = -\mathrm{sech} x \tanh x \qquad \frac{d}{dx}(\mathrm{csch}\ x) = -\mathrm{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \tan x \sec x\,dx = \sec x + C$$

$$\int \cot x \csc x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C,\ a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ |x| < |a|$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C,\ |x| > 1$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C,\ |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$